DETERMINACY EXERCISES
DAY 2

Problem 1. Show the following.

1. $|2^\omega| = |\omega^\omega| = |\mathbb{R}| = |\mathbb{R}\times\mathbb{R}|$.
2. $|\mathbb{R}^\omega| = |\mathbb{R}|$.
3. (Uses choice.) If X is infinite, then $|X| = |X \times X| = |X^{<\omega}|$.
4. (Uses choice.) κ^+ is regular, for all infinite cardinals κ.

Problem 2. Show AD implies $\mathsf{AC}_\omega(\mathbb{R})$.

Problem 3. Assume $\mathsf{AC}_\omega(\mathbb{R})$. Suppose $\{A_n\}_{n\in\omega}$ is a collection of non-empty countable sets of reals. Show $\bigcup_{n\in\omega} A_n$ is countable.

Problem 4. Show $\mathsf{AC}_\omega(\mathbb{R})$ is equivalent to the following seemingly weaker principle: For all countable collections $\{A_n\}_{n\in\omega}$ of non-empty sets of reals, there is some infinite $I \subseteq \omega$ and $f : I \to \bigcup_{n\in I} A_n$ with $f(n) \in A_n$ for every $n \in I$.

Problem 5. Let $\kappa > \omega$ be a regular cardinal. Show that if $\langle C_\xi\rangle_{\xi<\alpha}$ is a sequence of clubs in κ with $\alpha < \kappa$, then $\bigcap_{\xi<\alpha} C_\xi$ is club in κ.

Problem 6. Let $\langle A_\alpha\rangle_{\alpha<\kappa}$ be a sequence of subsets of a regular cardinal $\kappa > \omega$. The **diagonal intersection** of this sequence is the set

$$\triangle_{\alpha<\kappa} A_\alpha = \{\alpha < \kappa \mid \alpha \in \textstyle\bigcap_{\beta<\alpha} A_\beta\}.$$

Show the diagonal intersection of a sequence of clubs in κ is club in κ.

Definition. A set $S \subseteq \kappa$ is **stationary** if for every club C in κ, $S \cap C \neq \emptyset$.

Problem 7. Let S be a stationary subset of a regular cardinal $\kappa > \omega$. Show if $F : S \to \kappa$ is a function such that $F(\alpha) < \alpha$ for all $\alpha \in S$, then there is a stationary set $S' \subseteq S$ on which F is constant.

Problem 8. Let $A \subseteq \omega_1$. Define a game $G_c(A)$ played on ω_1 as follows: Players I and II take turns to produce a sequence of ordinals $\xi_0 < \xi_1 < \xi_2 < \ldots$ with each $\xi_n < \omega_1$. Player I wins if and only if $\sup_{n<\omega} \xi_n \in A$. Show the game $G_c(A)$ is determined if and only if A either contains or is disjoint from a club in ω_1.