DETERMINACY EXERCISES
DAY 3

Problem 1. Show ω^ω and 2^ω are complete metric spaces with the metric d defined in class.

Problem 2. Show that there are only countably many clopen subsets of 2^ω. Is the same true for Baire space, ω^ω?

Recall for topological spaces X, Y, $f : X \to Y$ is continuous if and only if $f^{-1}[U]$ is open in X for all open $U \subseteq Y$. Spaces X, Y are **homeomorphic** if there is a bijection $f : X \to Y$ so that both f and f^{-1} are continuous.

Problem 3. Let $f : \omega^\omega \to \omega^\omega$. Prove f is continuous if and only if whenever $f(x) = y$ and $n \in \omega$, there is some m such that $N_{x\restriction m} \subseteq f^{-1}[N_{y\restriction n}]$.

Problem 4. Let s_n be an enumeration of $\omega^{<\omega}$. Suppose $f : \omega^\omega \to 2^\omega$ is defined by setting $f(x)(n) = 0$ if and only if $x \in N_{s_n}$. Show f is continuous.

Problem 5. Show the following.

1. 2^ω and ω^ω are *not* homeomorphic: there is no continuous surjection $f : 2^\omega \to \omega^\omega$.
2. There exists a countable set $C \subseteq 2^\omega$ so that ω^ω and $2^\omega \setminus C$ (in the subspace topology) are homeomorphic.
3. ω^ω and $\mathbb{R} \setminus \mathbb{Q}$ are homeomorphic. (This justifies in part our nasty habit of calling elements of ω^ω "reals.")

Problem 6. Let $X \subseteq \omega^\omega$. Define X^* to be the collection

$$X^* = \{y \in \omega^\omega \mid (\exists x \in X)(\exists N \in \omega)(\forall n \geq N)x(n) = y(n)\}.$$

That is, X^* consists of all y that agree with some $x \in X$ on a tail end. Show the following.

1. If X is meager, then so is X^*.
2. If X is meager, then there is a nowhere dense set C such that $X \subseteq C^*$.