DETERMINACY EXERCISES
DAY 7

Let E be an equivalence relation on X. A function $f : X \to X$ is a **selector for** E if $f(x) = f(y)$ whenever $x \mathrel{E} y$, and $x \mathrel{E} f(x)$ for all $x \in X$.

Problem 1. Let E_0 be the equivalence relation on ω^ω given by setting $x \mathrel{E_0} y$ exactly when x and y are eventually equal. Prove that if $f : \omega^\omega \to \omega^\omega$ is a selector for E_0, then f is not Baire measurable.

Problem 2. Let $I = \{x \in \omega^\omega \mid \lim_{n\to\infty} x(n) = \infty\}$. Is I meager or comeager?

Problem 3. Let $C \subseteq \omega^\omega$ be clopen and $A \subseteq \omega^\omega$ any set with $A \neq \varnothing, \omega^\omega$. Describe a winning strategy for Player II in $G_{\mathrm{W}}(C, A)$.

Problem 4. Show, assuming the Axiom of Choice, that there exist sets $A, B \subseteq \omega^\omega$ so that $A \not\leq_{\mathrm{W}} B$ and $B \not\leq_{\mathrm{W}} \neg A$.

Definition. Let $A \subseteq \omega^\omega$ and $\mathbf{\Gamma}$ a pointclass closed under continuous substitution. We say A is **$\mathbf{\Gamma}$-complete** if $A \in \mathbf{\Gamma}$ and $B \leq_{\mathrm{W}} A$ for all $B \in \mathbf{\Gamma}$.

Problem 5. Let $A \subseteq \omega^\omega$ be a set consisting of just a single point. Show explicitly that A is $\mathbf{\Pi}^0_1$-complete, either by describing for each closed $C \subseteq \omega^\omega$ a continuous function $f : \omega^\omega \to \omega^\omega$ with $f^{-1}[A] = C$, or (equivalently) a winning strategy for Player II in $G_{\mathrm{W}}(C, A)$.

Problem 6. Let Q be the set of all $x \in \omega^\omega$ which are eventually constant. Show explicitly that Q is $\mathbf{\Sigma}^0_2$-complete.

Problem 7. Do the following.

1. Fix a bijection $\psi : \omega \times \omega \to \omega$. Let
$$P = \{x \in \omega^\omega \mid (\forall m)(\exists N)(\forall n > N) x(\psi(m, n)) = 0\}.$$
You can think of P as the set of $\omega \times \omega$ matrices with all rows eventually zero. Show that P is $\mathbf{\Pi}^0_3$-complete.
2. Using the previous item, show that the set
$$I = \{x \in \omega^\omega \mid \lim_{n\to\infty} x(n) = \infty\}$$
is $\mathbf{\Pi}^0_3$-complete.