DETERMINACY EXERCISES
DAY 8

PROBLEM 1. Let Q_0 be the set of all eventually zero members of ω^ω, and Q the set of all eventually constant members of ω^ω. Show $Q \leq_{\mathrm{L}} Q_0$ directly, e.g. by giving a winning strategy for Player II in $G_{\mathrm{L}}(Q, Q_0)$.

PROBLEM 2. Give an example of sets $A, B \subseteq \omega^\omega$ with $A \equiv_{\mathrm{W}} B$ but $A \not\equiv_{\mathrm{L}} B$.

PROBLEM 3. Assume AD. Show for all $A, B \subseteq \omega^\omega$ that $A <_{\mathrm{W}} B$ implies $A <_{\mathrm{L}} B$.

PROBLEM 4. Assume AD. Let $A \subseteq \omega^\omega$ and let $A^+ = \{\langle 0\rangle^\frown x \mid x \in A\}$. Show $A^+ \equiv_{\mathrm{L}} A$ if and only if $A \not\equiv_{\mathrm{L}} \neg A$.

PROBLEM 5. In this problem, we determine which Lipschitz degrees are self-dual in terms of how they appear in the Lipschitz hierarchy. You may assume AD throughout.

DEFINITION. Let a, c be Lipschitz degrees. We say c is the **L-successor** of a if $a <_{\mathrm{L}} c$ and whenever b is a Lipschitz degree with $a <_{\mathrm{L}} b$, we have $c \leq_{\mathrm{L}} b$.

If a is not an L-successor, we say a has **countable cofinality** if there is an ω-sequence $a_0 <_{\mathrm{L}} a_1 <_{\mathrm{L}} a_2 <_{\mathrm{L}} \ldots <_{\mathrm{L}} a$ so that a is the least degree above $\{a_n\}_{n\in\omega}$; otherwise we say a has **uncountable cofinality**.

1. Suppose $a = [A]_{\mathrm{L}}$ is a Lipschitz degree with $a \neq \neg a$. Let
$$C = A \oplus \neg A = \{\langle 0\rangle^\frown x \mid x \in A\} \cup \{\langle 1\rangle^\frown x \mid x \notin A\}.$$
Show $c = [C]_{\mathrm{L}}$ is self-dual, and is the L-successor of a.
2. Suppose $a = [A]_{\mathrm{L}}$ is a self-dual Lipschitz degree: $a = \neg a$. Let
$$C = A^+ = \{\langle 0\rangle^\frown x \mid x \in A\}.$$
Show $c = [C]_{\mathrm{L}}$ is self-dual, and is the L-successor of a.
3. Suppose a is a Lipschitz degree, but not an L-successor. Show that if a has countable cofinality, then a is self-dual.
4. Suppose a is a Lipschitz degree, not an L-successor, and a has uncountable cofinality. Show a is not self-dual: $a \neq \neg a$.

PROBLEM 6. Assume DC. Show Θ has uncountable cofinality: that is, there is no map $f : \omega \to \Theta$ with $f[\omega]$ unbounded in Θ.

PROBLEM 7. $\mathsf{Collection}$ asserts: Suppose $\{\mathcal{A}_x\}_{x\in\omega^\omega}$ is a family with $\varnothing \neq \mathcal{A}_x \subseteq \mathcal{P}(\mathbb{R})$ for all $x \in \omega^\omega$. Then there is a function $F : \omega^\omega \times \omega^\omega \to \mathcal{P}(\mathbb{R})$ such that for all $x \in \omega^\omega$, there is $y \in \omega^\omega$ with $F(x, y) \in \mathcal{A}_x$. (Note this is weaker than $\mathsf{AC}_{\mathbb{R}}(\mathcal{P}(\mathbb{R}))$; it says that from continuum many non-empty collections $\mathcal{A}_x$ of sets of reals, we can simultaneously narrow down each $\mathcal{A}_x$ to a set of size continuum.)

1. Assume $\mathsf{Collection}$ and show Θ is regular.
2. Assume $\mathsf{AD} + \mathsf{DC} +$ "Θ is regular". Prove $\mathsf{Collection}$.