DETERMINACY EXERCISES
DAY 9

Problem 1. Let X and Y be Polish spaces, and let $A \subseteq X \times Y$ belong to $\mathbf{\Sigma}^1_n$ for some $n \in \omega$. Show that

$$\exists^X A = \{y \in Y \mid (\exists x \in X)\langle x, y\rangle \in A\}$$

is in $\mathbf{\Sigma}^1_n$ as well.

Problem 2. Let X, Y be Polish spaces. Show the following:

1. If $f : X \to Y$ is continuous then its graph $\{\langle x, y\rangle \mid f(x) = y\}$ is closed in $X \times Y$.
2. If $A \subseteq X$ is $\mathbf{\Sigma}^1_1$ and $f : X \to Y$ is continuous then $f[A]$ is also $\mathbf{\Sigma}^1_1$.
3. A set $A \subseteq X$ is $\mathbf{\Sigma}^1_1$ if and only if $f[C] = A$ for some closed $C \subseteq \omega^\omega$ and continuous $f : \omega^\omega \to X$.
4. A set $A \subseteq X$ is $\mathbf{\Sigma}^1_1$ if and only if $f[\omega^\omega] = A$ for some continuous $f : \omega^\omega \to X$.

Problem 3. Let X, Y be Polish and suppose $f : X \to Y$ is Borel.

1. If you haven't already, show that the graph $\{\langle x, y\rangle \mid f(x) = y\}$ is Borel in $X \times Y$.
2. Show that if $A \subseteq X$ is $\mathbf{\Sigma}^1_1$, then $f[A]$ is also $\mathbf{\Sigma}^1_1$.

Problem 4. Let $I = \{x \in \omega^\omega \mid \lim_{n\to\infty} x(n) = \infty\}$. Give an explicit definition of a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ with $I = \exists^{\omega^\omega}[T]$; similarly for $\omega^\omega \setminus I$.

Definition. We say a pointclass $\mathbf{\Gamma}$ has the **separation property** if whenever $A, B \in \mathbf{\Gamma}$ with $A \cap B = \varnothing$, there exists a set $C \in \mathbf{\Gamma} \cap \neg\mathbf{\Gamma}$ so that $A \subseteq C$ and $B \cap C = \varnothing$. $\mathbf{\Gamma}$ has the **reduction property** if whenever $A, B \in \mathbf{\Gamma}$, we have some $A', B' \in \mathbf{\Gamma}$ with $A' \cup B' = A \cup B$, $A' \subseteq A$, $B' \subseteq B$, and $A' \cap B' = \varnothing$.

Problem 5. Show that if $\mathbf{\Gamma}$ has the reduction property then $\neg\mathbf{\Gamma}$ has the separation property.

Problem 6. In this problem we outline a game argument due to Blackwell to prove a strengthening of Lusin's separation theorem. Let S, T be trees in $\omega^{<\omega} \times \omega^{<\omega}$ so that $A = \exists^{\omega^\omega}[S]$ and $B = \exists^{\omega^\omega}[T]$. For each $z \in \omega^\omega$, define a game $G(z)$ as follows: Player I plays $x \in \omega^\omega$ and II plays $y \in \omega^\omega$.

I	$x(0)$		$x(1)$		$\dots$	$x(n)$		$\dots$
II		$y(0)$		$y(1)$	$\dots$		$y(n)$	$\dots$

Player I wins if for some $n \in \omega$, we have $\langle x \restriction n, z \restriction n\rangle \in S$ and $\langle y \restriction n, z \restriction n\rangle \notin T$; if for some n we have $\langle x \restriction n+1, z \restriction n+1\rangle \notin S$ while $\langle y \restriction n, z \restriction n\rangle \in T$, then Player II wins. If infinitely many moves are made and neither happens, then the play is a draw.

1. Explain why, for each $z \in \omega^\omega$, at least one of the players has a strategy to win or force a draw.
2. Use this to prove the following:

Theorem (Kuratowski). *Suppose $A, B \subseteq \omega^\omega$ are $\mathbf{\Sigma}^1_1$. Then there are sets A', B', also in $\mathbf{\Sigma}^1_1$, so that $A \subseteq A'$, $B \subseteq B'$, $A \cap B = A' \cap B'$, and $A' \cup B' = \omega^\omega$.*