DETERMINACY EXERCISES
DAY 10

PROBLEM 1. Fix an enumeration $\{s_i\}_{i\in\omega}$ of $\omega^{<\omega}$. For each $x \in 2^\omega$, let $T_x \subseteq \omega^{<\omega}$ be the set $\{s_i \mid x(i) = 1\}$. Locate the following sets in the Borel or projective hierarchies; that is, determine the optimal pointclass to which they belong.

1. $\{x \in 2^\omega \mid T_x \text{ is a tree}\}$;
2. $\{x \in 2^\omega \mid T_x \text{ is a well-founded tree}\}$;
3. $\{x \in 2^\omega \mid T_x \subseteq 2^{<\omega} \text{ is a well-founded tree}\}$.

PROBLEM 2. Let $L \subseteq \omega^\omega$ be the set

$$\{x \in (\omega \setminus \{0\})^\omega \mid \exists \langle k_n \rangle_{n\in\omega} \text{ increasing, such that for all } n,\ x(k_n) \text{ divides } x(k_{n+1})\}.$$

Show that L is $\mathbf{\Sigma}^1_1$-complete.

PROBLEM 3. Show for each $\gamma < \omega_1$ that $\mathrm{WO}_\gamma = \{x \in \mathrm{WO} \mid \mathrm{ot}(x) = \gamma\}$ is $\mathbf{\Sigma}^1_1$. Is it $\mathbf{\Delta}^1_1$?

PROBLEM 4. As in class, define for $x \in 2^\omega$ the binary relation R_x on ω so that $i \mathrel{R_x} j$ iff $x(\ulcorner i, j \urcorner) = 1$.

1. Let σ be a first order sentence in the language of one binary relation, R. Show $\{x \in 2^\omega \mid (\omega, R_x) \models \sigma\}$ is Borel.
2. Show the same with T a first order theory.

PROBLEM 5. Let $T \subseteq X^{<\omega} \times Y^{<\omega}$ be a tree; define the **projection**

$$p[T] = \{y \in Y^\omega \mid (\exists x \in X^\omega) \mid \langle x, y \rangle \in [T]\}.$$

We say a set $A \subseteq \omega^\omega$ is **κ-Suslin** if there is a tree $T \subseteq \kappa^{<\omega} \times \omega^{<\omega}$ with $p[T] = A$.

Recall that a tree $T \subseteq \omega^{<\omega}$ has $[T] = \varnothing$ if and only if there is a rank function $\rho : T \to \mathrm{ON}$.

1. Show that if $T \subseteq \omega^{<\omega}$ has a rank function ρ, then we can assume $\rho : T \to \omega_1$.
2. Use this to show that if A is $\mathbf{\Pi}^1_1$, then A is ω_1-Suslin.
3. Show that if A is $\mathbf{\Sigma}^1_2$, then A is ω_1-Suslin.