DETERMINACY EXERCISES
DAY 12

Definition. Recall $\mathcal{D}_T$ is the collection of Turing degrees. A set $A \subseteq \omega^\omega$ is called **Turing invariant** if it is a union of Turing degrees. **Turing determinacy** is the statement that whenever $A \subseteq \omega^\omega$ is Turing invariant, the game $G(A)$ is determined.

Problem 1 (Martin). Assume Turing determinacy. Show every set of Turing degrees either contains, or is disjoint from, a cone.

Problem 2 (Erdös-Rado). (Uses choice.) Let κ be an infinite cardinal. Show there is a coloring $c : [\kappa]^\omega \to 2$ of the collection of countably infinite subsets of κ so that there is no infinite c-homogeneous subset of κ.

Problem 3. In this exercise we use a determinacy argument to show that $\boldsymbol{\Sigma}^1_1$ sets have the perfect set property. Let $F \subseteq \omega^\omega \times \omega^\omega$ and let $A = \exists^{\omega^\omega} F$. The **unfolded perfect set game** $G^*_{\mathrm{PS}}(F)$ is played as follows:

I	$x(0), s^0_0, s^1_0$		$x(1), s^0_1, s^1_1,$		$\dots$	$x(n), s^0_n, s^1_n$		$\dots$
II		i_0		i_1	$\dots$		i_n	$\dots$

Each $x(n) \in \omega$, $s^i_n \in \omega^{<\omega}$, and $i_n \in \{0,1\}$. The rules: Player I plays $x(0) \in \omega$ and s^0_0, s^1_0 with $s^0_0 \perp s^1_0$. Player II plays $i_n \in \{0,1\}$. Having fixed $s^{i_n}_n$, Player I must choose incompatible extensions s^0_{n+1}, s^1_{n+1} of $s^{i_n}_n$; that is, $s^0_{n+1}, s^1_{n+1} \supsetneq s^{i_n}_n$, and $s^0_{n+1} \perp s^1_{n+1}$.

After infinitely many rounds, set $y = \bigcup_{n\in\omega} s^{i_n}_n$. Then Player I wins if and only if $\langle x, y\rangle \in F$. (In particular, if Player I wins then $y \in A$.)

1. Show that if Player I has a winning strategy in $G^*_{\mathrm{PS}}(F)$, then A contains a non-empty perfect set. (The converse can fail!)
2. Prove that Player II has a winning strategy in $G^*_{\mathrm{PS}}(F)$ if and only if A is countable.
3. Deduce that if $\boldsymbol{\Gamma}$ is a pointclass closed under continuous substitution and $\boldsymbol{\Gamma}$-DET holds, then every set in $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ has the perfect set property. In particular, by the Gale-Stewart theorem, all $\boldsymbol{\Sigma}^1_1$ sets have the perfect set property; and if there is a measurable cardinal, then all $\boldsymbol{\Sigma}^1_2$ sets have the perfect set property.

Problem 4. Show that all $\boldsymbol{\Sigma}^1_1$ subsets of ω^ω have the Baire property.

Problem 5. Let $S^1 \subset \mathbb{R}^2$ be the unit circle, and fix an irrational number γ. Let D be the orbit of $(1,0)$ under iterated rotation of S^1 by $\gamma\pi$; that is,

$$D = \{(\cos(k\gamma\pi), \sin(k\gamma\pi)) \in S^1 \mid k \in \mathbb{Z}\}.$$

Show D is dense in S^1.

Problem 6. Let $S_\infty = \{f \in \omega^\omega \mid f \text{ is a bijection}\}$. Recall S_∞ is a group with multiplication given by function composition.

1. Show S_∞ with the subspace topology inherited from ω^ω is a Polish space.
2. Show there is a sequence $\langle g_n\rangle_{n\in\omega}$ in S_∞ so that $\{g_n\}_{n\in\omega}$ is a dense subset of S_∞, and generates a free subgroup of S_∞.