DETERMINACY EXERCISES
DAY 13

PROBLEM 1. For each real x, define

$$\omega_1^x = \sup\{\text{ot}(y) \mid y \in \text{WO}, y \leq_T x\}.$$

So ω_1^x is the least ordinal not computable from x.

Let $\mu \subseteq \mathcal{P}(\omega_1)$ be the set of X so that $\{[x]_T \mid \omega_1^x \in X\}$ contains a cone. Show, assuming Turing determinacy, that μ is a normal measure on ω_1.

PROBLEM 2. Assume AD. Show Θ is a limit cardinal.

PROBLEM 3. Let G be the graph on 2^ω with $x \, G \, y$ iff $|\{n \in \omega \mid x(n) \neq y(n)\}| = 1$. Show $\chi(G) = 2$, but $\chi_B(G) > 2$—indeed, $\chi_B(G) > \omega$.

PROBLEM 4. Let $F : 2^\omega \to 2^\omega$ be the **odometer map**, defined by setting

$$F(x)(n) = \begin{cases} 1 - x(n) & \text{if } x(i) = 1 \text{ for all } i < n; \\ x(n) & \text{otherwise.} \end{cases}$$

In other words, to get $F(x)$, add 1 to $x(0)$ (mod 2), and carry. Let $G = G_{\{F\}}$.

1. Show the connected components of G are precisely the E_0 classes, except for one which contains all eventually constant sequences.
2. Show $\chi_B(G) = \chi(G) = 2$.

PROBLEM 5. Let S^1 be the unit circle, $S^1 = \{e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$. Fix an irrational $\gamma \in \mathbb{R}$ and put $g = e^{i\gamma\pi}$. Let G be the graph on S^1 obtained by setting $x \, G \, y$ iff $x = g \cdot y$ or $y = g \cdot x$ (that is, two points on the circle are G-adjacent if one is sent to the other by a rotation of S^1 by $\gamma\pi$ radians.)

Show G is acyclic with degree 2, and so $\chi(G) = 2$; but $\chi_B(G) \geq 3$.

PROBLEM 6. Fix $\gamma \in \mathbb{R} \setminus \mathbb{Q}$. Let G be the graph on $\mathbb{R}$ obtained by setting $x \, G \, y$ iff $|x - y| = \gamma$. Show $\chi_B(G) = 2$.

PROBLEM 7. Suppose G is a d-regular Borel graph on a Polish space X. Prove (without appealing to anything we haven't proved in class!) that there are Borel functions $F_i : X \to X$ with $i < d$ so that $G = G_{\{F_i\}}$.