DETERMINACY EXERCISES
WEEKEND 1

Recall that a set $P \subseteq X$ in a Polish space is **perfect** if it is closed, and every $x \in P$ is a limit point of P.

Problem 1. If P is perfect and non-empty, then $|P| = \mathfrak{c}$.

Definition. A set $A \subseteq X$ has the **perfect set property** if either A is countable, or A contains a non-empty perfect subset.

Problem 2 (Cantor-Bendixon). Suppose $K \subseteq X$ is closed, with X a Polish space. Inductively define

- $K_0 = K$,
- $K_{\alpha+1} = K'_\alpha = \{x \in K_\alpha \mid x \text{ is a limit point of } K_\alpha\}$, and
- $K_\lambda = \bigcap_{\alpha<\lambda} K_\alpha$ for limit ordinals λ.

Show there exists $\alpha < \omega_1$ so that $K_{\alpha+1} = K_\alpha$, and that this K_α is perfect.

Conclude $K = P \cup C$ with P perfect and C countable. In particular, K has the perfect set property.

Problem 3. Show, using the Axiom of Choice, that there is a set without the perfect set property.

Fix $A \subseteq \omega^\omega$. The **perfect set game** $G_{\mathrm{PS}}(A)$ is played as follows:

$$\begin{array}{c|cccccccc} \mathrm{I} & s_0^0, s_0^1 & & s_1^0, s_1^1 & & \dots & s_n^0, s_n^1 & & \dots \\ \hline \mathrm{II} & & i_0 & & i_1 & \dots & & i_n & \dots \end{array}$$

Each $s_n^i \in \omega^{<\omega}$ and $i_n \in \omega$. Here are the rules: Player I plays s_0^0, s_0^1 with $s_0^0 \perp s_0^1$. Player II plays $i_n \in \{0,1\}$. Having fixed $s_n^{i_n}$, Player I must choose incompatible extensions s_{n+1}^0, s_{n+1}^1 of $s_n^{i_n}$; that is, $s_{n+1}^0, s_{n+1}^1 \supsetneq s_n^{i_n}$, and $s_{n+1}^0 \perp s_{n+1}^1$.

A play of this game produces $x = \bigcup_{n\in\omega} s_n^{i_n}$. Player I wins if and only if $x \in A$.

Problem 4. Show Player I has a winning strategy in $G_{\mathrm{PS}}(A)$ if and only if A contains a non-empty perfect subset.

Problem 5. Show Player II has a winning strategy in $G_{\mathrm{PS}}(A)$ if and only if A is countable.

Problem 6. Show that if $\mathbf{\Gamma}$ is closed under continuous substitution, then $\mathbf{\Gamma}$-DET implies every $A \in \mathbf{\Gamma}(\omega^\omega)$ has the perfect set property; in particular, AD implies that every set in ω^ω has the perfect set property.

Problem 7. Show AD implies there is no injection $f : \omega_1 \to \omega^\omega$.

Problem 8. Show (without choice) that AD_{ω_1} is false.