FORCING EXERCISES
DAY 5

PROBLEM 1. Recall we say a set $\mathcal{X}$ generates an ultrafilter on X if the smallest filter containing $\mathcal{X}$ is an ultrafilter. We define the **ultrafilter number** $\mathfrak{u}$ to be the smallest size of a generating set for a *nonprincipal* ultrafilter on ω.

- Show that $\omega < \mathfrak{u} \leq 2^\omega$.
- Show that MA implies $\mathfrak{u} = 2^\omega$.

PROBLEM 2. Let $\mathcal{F}$ be a filter on ω and let $\langle a_n \mid n < \omega \rangle$ be a sequence of real numbers. Define $\lim^{\mathcal{F}} a_n = a$ if for every ϵ there is a set $A \in \mathcal{F}$ such that $|a - a_n| < \epsilon$ for all $n \in A$.

1. For which filter $\mathcal{F}$ does this notion of convergence coincide with the usual one?
2. Suppose that $\mathcal{U}$ is a nonprincipal ultrafilter on ω and that $\langle a_n \mid n < \omega \rangle$ is a bounded sequence of real numbers. Show that there is a unique $a \in \mathbb{R}$ such that $\lim^{\mathcal{U}} a_n = a$.

For the remainder of the problems and definitions κ is a regular cardinal.

DEFINITION 1. A set $C \subseteq \kappa$ is **closed** if for all $\alpha < \kappa$, if $C \cap \alpha$ is unbounded in α, then $\alpha \in C$. A set $C \subseteq \kappa$ is **club** if it is closed and unbounded in κ.

PROBLEM 3. Let $\mathcal{C}$ be the collection of subsets of κ that contain a club. Show $\mathcal{C}$ is a filter on κ. We call this filter the **club filter** on κ.

DEFINITION 2. Let λ be a regular cardinal. We say that a filter $\mathcal{F}$ is **λ-complete** if it is closed under intersections of size less than λ.

PROBLEM 4. Show that the club filter on κ is κ-complete.

DEFINITION 3. $\mathcal{F}$ be a filter on κ. We say that $\mathcal{F}$ is **normal** if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of elements of $\mathcal{F}$, the set

$$\triangle_{\alpha<\kappa} A_\alpha = \{\alpha < \kappa \mid \alpha \in \textstyle\bigcap_{\beta<\alpha} A_\beta\}$$

is in $\mathcal{F}$. This set is called the **diagonal intersection** of the sets A_α.

PROBLEM 5. Show that the club filter on κ is normal.

DEFINITION 4. A set $S \subseteq \kappa$ is **stationary** if for every club C in κ, $S \cap C \neq \emptyset$.

PROBLEM 6. Let S be a stationary subset of κ. Show that for every function $F : S \to \kappa$ such that for all $\alpha \in S$ $F(\alpha) < \alpha$, there is a stationary set $S' \subseteq S$ on which F is constant.

PROBLEM 7 (*). Show that if $\mathcal{U}$ is a Ramsey ultrafilter, then any $\chi : [\omega]^k \to l$ has a monochromatic set in $\mathcal{U}$.