FORCING EXERCISES
DAY 6

PROBLEM 1. Show that any two countable dense linear orders without endpoints are isomorphic.

PROBLEM 2. Let κ be a regular cardinal. Construct a tree of height κ with no cofinal branch.

DEFINITION 1. A tree T is **well-pruned** if it has a unique $<_T$-least element (that is, $|\operatorname{Lev}_0(T)| = 1$) and for all $\alpha < \beta < \operatorname{ht}(T)$ and for all $x \in \operatorname{Lev}_\alpha(T)$, there is a $y \in \operatorname{Lev}_\beta(T)$ with $x <_T y$.

PROBLEM 3. Let κ be a regular cardinal. Construct a well-pruned tree of height κ with no cofinal branch.

PROBLEM 4. Let κ be a regular cardinal. Show that every κ-tree has a well-pruned subtree.

PROBLEM 5 (*). Recall we said an ω_1-tree $(T, <_T)$ is **normal** if

1. T is well-pruned,
2. For every $x \in T$, the set of immediate $<_T$-successors of x is infinite,
3. If $\lambda < \omega_1$ is a limit ordinal and $x, y \in \operatorname{Lev}_\lambda(T)$ have the same predecessors, then $x = y$.

Show that if there is a Suslin tree, then there is a normal Suslin tree.

DEFINITION 2. Let $(T, <_T)$ be a partially ordered set. T is **splitting** if for every $x \in T$ there are $y_0, y_1 \in T$ such that $x <_T y_0$, $x <_T y_1$, and there is no $z \in T$ such that $y_0, y_1 \leq_T z$.

PROBLEM 6. Let κ be a regular cardinal. Suppose that T is a well-pruned κ-tree with no cofinal branch. Show that T is splitting.

PROBLEM 7. Show that if T is a special ω_1-tree, then T has no cofinal branch.

PROBLEM 8. Show that if T is a special ω_1-tree, then T is not Suslin.

PROBLEM 9 (*). Let κ be an uncountable cardinal. Suppose that T is a tree of height κ^+ with levels of size less than κ. Show that T has a cofinal branch.

THEOREM 1 (Finite Ramsey Theorem). *For every $k < \omega$ there is $n < \omega$ such that for every $\chi : [n]^2 \to 2$ there is a monochromatic set of size k.*

PROBLEM 10. Use the König Infinity Lemma and the infinite Ramsey Theorem to prove the finite Ramsey Theorem.