FORCING EXERCISES
DAY 8

Problem 1. Prove Proposition 9.3 from the notes.

Problem 2. The Axiom Scheme of Collection asserts that for any definable property $\phi(u, v)$ and set a, if for every $x \in a$ there is y so that $\phi(x, y)$ holds, then there is a set b so that every $x \in a$ has such a $y \in b$. Formally,

$$\forall a[((\forall x \in a)\exists y\phi(x, y)) \to \exists b(\forall x \in a)(\exists y \in b)\phi(x, y)].$$

(Note that unlike Replacement, we don't assume the y above is unique!) Prove that Collection follows from the axioms of ZF.

Problem 3 (*). Show that "x is a well-founded relation" is absolute for transitive models of ZFC.

Problem 4. Let X be a countable elementary substructure of some H_θ where θ is regular and uncountable. Note that X is not transitive!

a. Suppose that $A \in X$ and $H_\theta \vDash$ "A is countable". Show that then $X \vDash$ "A is countable" and $A \subseteq X$.
b. Show that $X \cap \omega_1 \in \omega_1$.
c. Define a countable set of ordinals that is not a member of X.
d. Show that $\omega_1 \in X$ if $\theta > \omega_1$.
e. Define a subset of ω which is not a member of X.

Problem 5. Let $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of elementary substructures of H_θ for some regular uncountable θ such that $X_\alpha \in X_{\alpha+1}$ for all $\alpha < \omega_1$ and $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$ for all limit γ. Show the following:

a. For all $\alpha < \beta < \omega_1$, $X_\alpha \prec X_\beta$.
b. $\{X_\alpha \cap \omega_1 \mid \alpha < \omega_1\}$ is club in ω_1.
c. There is a club C in ω_1 such that for all $\gamma \in C$, $X_\gamma \cap \omega_1 = \gamma$.

Problem 6. A cardinal is called **inaccessible** if it's a regular limit cardinal; it's **strongly inaccessible** if in addition, $2^\lambda < \kappa$, for all $\lambda < \kappa$.

a. Show that if κ is strongly inaccessible, then $V_\kappa \models$ ZFC.
b. Does $V_\kappa \models$ ZFC imply κ is an inaccessible cardinal?
c. Show ZFC $\not\vdash (\exists\alpha)V_\alpha \models$ ZFC.

Problem 7 (*). A theory T is **finitely axiomatizable** if there is a finite $\Sigma \subseteq T$ so that $\Sigma \vdash \phi$, for every sentence $\phi \in T$. Show ZFC is not finitely axiomatizable.

Problem 8 (*). Is there a singular cardinal κ so that H_κ satisfies Replacement?