FORCING EXERCISES
DAY 9

Definition 1. A formula in the language of set theory is a **Σ_1-formula** if it is of the form $\exists u \psi(u, v)$, where ψ is a Δ_0-formula. A **Π_1-formula** is one of the form $\forall u \psi(u, v)$. We say a formula $\phi(v)$ is a **Δ_1-formula** if ZFC $\vdash \forall v(\phi(v) \leftrightarrow \sigma(v))$ and ZFC $\vdash \forall v(\phi(v) \leftrightarrow \pi(v))$, where σ is Σ_1 and π is Π_1.

Problem 1. Show Δ_1-formulas are absolute for transitive models of ZFC.

Problem 2. Show the relations "$\mathrm{rk}(x) = \alpha$", "σ is a $\mathbb{P}$-name", "$\rho(\sigma) = \alpha$" are all expressible by Δ_1-formulas.

Problem 3. Let τ be a signature, and fix a set X with U an ultrafilter on X. Suppose for each $i \in X$ that $\mathcal{M}_i$ is a τ-structure with non-empty domain M_i. We will define a new τ-structure, the **ultraproduct of** $\{\mathcal{M}_i\}_{i \in X}$, as follows.

Let $\Pi_{i \in X} M_i$ be the collection of functions $f : X \to M_i$ such that $f(i) \in M_i$ for all i. For $f, g \in \Pi_{i \in X} M_i$, define $f =_U g$ if and only if $\{i \in X \mid f(i) = g(i)\} \in U$.

1. Show $=_U$ is an equivalence relation.
2. Let $\Pi_U M_i$ be the set of equivalence classes $[f]$ of $=_U$. We make $\Pi_U M_i$ the domain of a τ-structure $\mathcal{M}$ as follows:
   - for constants c of τ, $c^{\mathcal{M}} = [i \mapsto c^{\mathcal{M}_i}]$;
   - for function symbols g, $g^{\mathcal{M}}([f_1], \ldots, [f_n]) = [i \mapsto g(f_1(i), \ldots, f_n(i))]$;
   - for relation symbols R, $R^{\mathcal{M}}([f_1], \ldots, [f_n])$ holds if and only if the set $\{i \mid R^{\mathcal{M}_i}(f_1(i), \ldots, f_n(i))\} \in U$.

   Show the above interpretations are well-defined, and that $\mathcal{M}$ so defined is a τ-structure.
3. Prove, for all formulas ϕ and functions $f_1, \ldots, f_n$, that

$$\mathcal{M} \models \phi([f_1], \ldots, [f_n]) \text{ if and only if } \{i \in X \mid \mathcal{M}_i \models \phi(f_1(i), \ldots, f_n(i))\} \in U.$$

Problem 4. Use the construction of the previous exercise to prove the Compactness theorem.

For the following problems, let M be a countable transitive model of ZFC, and let $\mathbb{P} \in M$ be a poset.

Problem 5. Suppose $\mathbb{P}$ is non-atomic. Show there are continuum many $\mathbb{P}$-generic filters over M.

Problem 6. Suppose $\mathbb{P}$ is not non-atomic. Show there is an $\mathbb{P}$-generic filter G in M.

Problem 7. Show G is an M-generic filter if and only if G meets every dense subset of $\mathbb{P}$ in M, is closed upwards, and any two conditions in G are compatible.

Problem 8. Suppose $A \subseteq \mathbb{P}$ is a maximal antichain in M, and that τ_p is a $\mathbb{P}$-name for each $p \in A$. Let $\pi = \{\langle \tau_p, p \rangle \mid p \in A\}$. What is the cardinality of $\pi[G]$?