FORCING EXERCISES
DAY 11

Problem 1. Let $\mathbb{P} \in M$ be a poset, and let $p \in \mathbb{P}$. We say that a set D is **dense below** p if for every $q \leq p$ there is an $r \leq q$ with $r \in D$.

1. Show that if G is $\mathbb{P}$-generic and D is dense below $p \in G$, then $G \cap D \neq \emptyset$.
2. Let $\mathbb{P} \in M$ and $p \in \mathbb{P}$. Let ϕ be a formula of the forcing language. Suppose that the set $D = \{q \in \mathbb{P} : q \Vdash \phi\}$ is dense below p. Show that $p \in D$.

Problem 2. Suppose G is $\mathbb{P}$-generic over M, where $\mathbb{P}$ is countable (in M). Show that if A is an uncountable set of ordinals in $M[G]$, then there is $B \subseteq A$, also uncountable (in $M[G]$), with $B \in M$.

Recall that a filter G is $\mathbb{P}$-generic over M if and only if it intersects every maximal antichain of $\mathbb{P}$ that belongs to M. The following fact is very important!

Problem 3 (Maximality of the Forcing Language). Show that whenever $p \Vdash (\exists x)\phi(x, \tau_1, \ldots, \tau_n)$, there is a name $\sigma \in M^{\mathbb{P}}$ such that $p \Vdash \phi(\sigma, \tau_1, \ldots, \tau_n)$.

Problem 4. An **automorphism** of a poset $\mathbb{P}$ is a bijection $i : \mathbb{P} \to \mathbb{P}$ so that $p \leq q$ if and only if $i(p) \leq i(q)$. An i of $\mathbb{P}$ in M induces a permutation of $M^{\mathbb{P}}$, which we also call i, by the following recursive definition for $\mathbb{P}$-names τ:

$$i(\tau) = \{\langle i(\sigma), i(p)\rangle : \langle \sigma, p\rangle \in \tau\}.$$

1. Show that if $i : \mathbb{P} \to \mathbb{P}$ is an automorphism, then $i(\check{x}) = \check{x}$ for $x \in M$.
2. Let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Prove $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ iff $i(p) \Vdash \phi(i(\tau_1), \ldots, i(\tau_n))$.

Problem 5. The forcing $\mathbb{C}$ of finite partial functions from ω to ω ordered by extension is commonly referred to as **Cohen forcing**. A **Cohen real over** M is a function $c : \omega \to \omega$ such that the corresponding filter $G = \{p \in \mathbb{C} : p \subseteq c\}$ is $\mathbb{C}$-generic over M.

1. Show that if G is a $\mathbb{C}$-generic filter over M then the function $g = \bigcup G$ is a Cohen real over M. (So Cohen generic filters and Cohen generic reals are essentially the same thing).
2. Suppose that c is a Cohen real over M. Show that the function mapping n to $c(2n)$ is also a Cohen real over M.
3. More generally suppose that $A \subseteq \omega$ is any infinite subset *belonging to* M. Let e_A enumerate A in increasing order. Show that the function mapping n to $c(e_A(n))$ is a Cohen real over M.
4. Give an example of an infinite $A \subseteq \omega$ belonging to $M[c]$ such that the function mapping n to $c(e_A(n))$ is not a Cohen real over M.

Problem 6. Let $\mathbb{B}$ be the poset of finite partial functions from ω to 2.

1. Let G be $\mathbb{C}$-generic over M. Show that $M[G]$ contains a filter H which is $\mathbb{B}$-generic over M. (Hint: generic filters are essentially reals).
2. Let G be $\mathbb{B}$-generic over M. Show $M[G]$ contains a Cohen real over M.

PROBLEM 7 (*). Let M a countable transitive model. Show there are Cohen reals c, d over M that are not both contained in any generic extension: indeed, so that no transitive $N \supset M \cup \{c, d\}$ with $N \cap \mathrm{On} = M \cap \mathrm{On}$ is a model of ZFC.