FORCING EXERCISES
FOR THE WEEKEND

Here is a collection of problems for the weekend. It's a good idea before you start these to make sure you understand the problems from the week you didn't get. In particular, some of these are much harder without assuming results from those exercises!

Problem 1. Show that a tree T of height ω has an infinite branch if and only if there is no function $f : T \to \mathrm{On}$ so that $f(x) > f(y)$ whenever $x <_T y$.

Problem 2. Construct an Aronszajn tree. Show the tree you construct is special.

Problem 3. Say a set $R \subseteq \omega_1 \times \omega$ is a *large rectangle* if it is of the form $A \times B$ where A is uncountable and B is infinite. Show that $\mathrm{MA}(\aleph_1)$ implies that every $S \subseteq \omega_1 \times \omega$ either contains a large rectangle or is disjoint from one.

Problem 4 (*). Assume $\mathrm{MA}(\aleph_1)$ and show all ccc posets are $\aleph_1$-Knaster.

Definition 1. Given a poset $\mathbb{P}$, $\mathrm{cc}(\mathbb{P})$ is the least cardinal κ such that $\mathbb{P}$ has no antichains of size κ.

Problem 5 (*). Suppose that $\mathbb{P}$ is a poset such that $\mathrm{cc}(\mathbb{P}) > n$ for all $n < \omega$. Show that $\mathrm{cc}(\mathbb{P}) > \omega$, i.e., show that $\mathbb{P}$ has an infinite antichain.

Problem 6 (**). Let $\mathbb{P}$ be a fixed poset. Show that $\mathrm{cc}(\mathbb{P})$ is a regular cardinal.