FORCING EXERCISES
FOR THE SECOND WEEKEND

Problem 1. Suppose $p \Vdash \phi$ and ZFC $\vdash (\phi \to \psi)$. Then $p \Vdash \psi$. (You can assume $M[G]$ is a model of ZFC).

Problem 2. Do the following:

1. Show that for any formulas ϕ and ψ and $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$ and $\sigma_1, \dots, \sigma_m \in M^{\mathbb{P}}$ we have that $p \Vdash \phi(\tau_1, \dots, \tau_n) \wedge \psi(\sigma_1, \dots, \sigma_m)$ if and only if both $p \Vdash \phi(\tau_1, \dots, \tau_n)$ and $p \Vdash \psi(\tau_1, \dots, \tau_m)$.
2. Show that for any formula ϕ and $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$, $p \Vdash \neg\phi(\tau_1, \dots, \tau_n)$ if and only if there is no $q \leq p$ such that $q \Vdash \phi$.

Problem 3. Suppose X is a proper elementary substructure of some H_θ, where $\theta > \omega_1$ is regular. Let M be the transitive collapse of X, with $\pi : M \to X \prec H_\theta$ the inverse of the collapse map. The least ordinal α so that $\pi(\alpha) \neq \alpha$ is called the **critical point** of π, $\mathrm{crit}(\pi)$. Let κ be the image of the critical point, $\kappa = \pi(\mathrm{crit}(\pi))$.

1. Show κ is a regular cardinal.
2. Suppose $C \in X$ is a club in κ. Show $X \cap \kappa \in C$.
3. Suppose $\kappa > \omega_1$. Show there is a stationary $S \in X$ with $X \cap \kappa \notin S$.

Definition 1. Let $\mathcal{M}$ be a structure, and fix an ultrafilter $\mathcal{U}$ on a set X. We define the **ultrapower of $\mathcal{M}$ by $\mathcal{U}$**, $\mathrm{Ult}(\mathcal{M},\mathcal{U})$, to be the ultraproduct of $\{\mathcal{M}_i\}_{i \in \mathcal{U}}$, where every $\mathcal{M}_i$ is $\mathcal{M}$. The **ultrapower embedding** is the map $j_{\mathcal{U}} : \mathcal{M} \to \mathrm{Ult}(\mathcal{M},\mathcal{U})$ that takes an element $a \in M$ to the constant function with value a, $j_{\mathcal{U}}(a) = [i \mapsto a]$.

Problem 4. Show $j_{\mathcal{U}}$ is an elementary embedding.

Problem 5. Suppose that V_δ is a model of ZFC, and that $\mathcal{U} \in V_\delta$ is a nonprincipal **countably complete** ultrafilter; that is, if $\{A_i\}_{i \in \omega}$ is a collection of sets in $\mathcal{U}$, then $\bigcap_{i \in \omega} A_i \in \mathcal{U}$.

1. Show that $\in$ as interpreted in $\mathrm{Ult}(V_\delta,\mathcal{U})$ is a well-founded relation.
   In light of this, we identify $\mathrm{Ult}(V_\delta,\mathcal{U})$ with its transitive collapse, so that $j_{\mathcal{U}} : V_\delta \to \mathrm{Ult}(V_\delta,\mathcal{U})$ is a map between transitive sets.
2. Show there is an ordinal $\kappa < \delta$ so that $j_{\mathcal{U}}(\kappa) \neq \kappa$. Prove the least such, $\mathrm{crit}(j_{\mathcal{U}})$, is a strongly inaccessible cardinal.
3. (*) Show V_δ and $\mathrm{Ult}(V_\delta,\mathcal{U})$ have the same ordinals, but are not the same set.
4. (*) Show there is a nonprincipal *normal* ultrafilter μ on $\kappa = \mathrm{crit}(j_{\mathcal{U}})$. Further show that $\mathrm{crit}(j_\mu) = \kappa = [\mathrm{id}]_\mu \in \mathrm{Ult}(V_\delta, \mu)$, where $\mathrm{id} : \kappa \to \kappa$ is the identity.

A cardinal bearing a normal ultrafilter is said to be **measurable**.