Math 215 - Review for Final

Due by 4pm, Monday, Dec 10

1. Give a truth table for each of the following propositional forms.

(a) $\neg(P \to Q)$

(b) $P \vee (Q \wedge P)$

(c) $P \to (P \leftrightarrow Q)$

(d) $(P \to Q) \wedge \neg(Q \to R)$

2. Give a useful denial of each of the following (the denials should not simply be the statement with a "not" in front):

(a) Every person who appreciates action movies loves Arnold Schwarzenegger.

(b) I'll buy dinner for anybody who can prove the Collatz conjecture.

(c) If the litter box is dirty, then it was the cat.

3. For each of the following statements, provide a negation in a form so that all quantifiers appear first; then decide which of the given statement and its negation is true, proving your answer.

(a) $\forall m \in \mathbb{N}\ m \nmid m+1$

(b) $\exists k \in \mathbb{N}\ \forall m \in \mathbb{N}\ k > 1 \wedge (0 < m \leq k \to m \mid k)$

(c) $\forall n \in \mathbb{N}\ \forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ y \geq x \wedge n \mid y$

4. For each of the following, state both the contrapositive and the converse. For both statements (two in each part), say whether it is true or false.

(a) $(n \in \mathbb{N}.)$ If n is prime, then n is a sum of two squares.

(b) $(f \in \text{Fun}(\mathbb{R}, \mathbb{R}).)$ If $\text{Im}(f) \subseteq \mathbb{Z}$, then f is constant.

(c) $(r \in \mathbb{Q}.)$ If $r = y^2$ for some $r \in \mathbb{R}$, then $r > 0$.

5. Show that if $a, b, c \in \mathbb{Z}$ and a divides both b and c, then a divides $b + c$.

6. Show by induction on n that for all positive naturals n,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

7. Show by induction on n that for all naturals n, if $n \geq 4$, then $2^n < n!$.

8. Show by example that for sets A, B, $A - B = \emptyset$ does not imply $A = B$.

9. Suppose $f : A \to B$ is a bijection. Show there is a bijection $F : \mathcal{P}(A) \to \mathcal{P}(B)$.

10. Suppose $g : A \to B$ is injective. Show there is a surjection $f : B \to A$.

11. For sets $A \subseteq \mathbb{N}$, define $2A := \{2n \mid n \in A\}$.

(a) Show the function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by $f(A) = 2A$ is one-to-one, but not onto.

(b) What is $|\mathbb{N}_{15} \cap 2\mathbb{N}_{10}|$?

12. Show that $\{x^3 - x \mid x \in \mathbb{N}, x \geq 1\} \subseteq \{n \in \mathbb{N} \mid 6 \text{ divides } n\}$. Does the reverse inclusion hold?

13. How many numbers between 200 and 900 (inclusive) are divisible by 5 or 9? Explain.

14. Suppose you roll two standard six-sided dice and take the difference of their outcomes. What are the probabilities of each of the differences 1,2,3,4,5? Explain.

15. How many binary sequences of length 10 do not contain two 1's in a row? Explain how you found your answer.

16. Define the set $\mathcal{P}_{12}(\mathbb{N}_{20})$, and show $|\mathcal{P}_{12}(\mathbb{N}_{20})| = |\mathcal{P}_8(\mathbb{N}_{20})|$.

17. For sets X, let $X \sim Y$ if and only if there is a bijection $f : X \to Y$. Show $\sim$ is an equivalence relation.

18. Give a function $f : \text{Fun}(\mathbb{N}, \mathbb{N}) \to \text{Fun}(\mathbb{N}, \{0, 1\})$ that is injective, and prove it is injective.

For the remaining problems, let n be a natural number. Define a relation $\equiv_n$ on $\mathbb{Z}$ by: $a \equiv_n b$ iff $n|(a - b)$.

19. Show $\equiv_n$ is an equivalence relation.

20. Show $\mathbb{Z}/\equiv_0$ is infinite, and that $|\mathbb{Z}/\equiv_1| = 1$. How many $\equiv_n$-equivalence classes are there, for $n \geq 2$?

21. Define an operation $\oplus$ on $\mathbb{Z}/\equiv_n$ by: $[a] \oplus [b] = [a + b]$. Show this operation is well-defined.

22. Define an operation $\otimes$ on $\mathbb{Z}/\equiv_n$ by: $[a] \otimes [b] = [ab]$. Show this operation is well-defined.

23. Suppose we define an "exponentiation" operation on $\mathbb{Z}/\equiv_4$ by: $\exp([m], [n]) = [m^n]$. Prove this operation is *not* well-defined.