Math 215 - Homework 2

Due Friday, September 21

1. For each of the following, give the statement's contrapositive and its converse. For each, either state that it is true or give a counterexample to show it is not.

(a) If n is an even integer, then n is not prime.

Contrapositive: If n is prime, then n is not an even integer.

Converse: If n is not prime, then n is an even integer.

Both statements are false: $n = 2$ is both prime and even; and 15 is not prime, but also odd.

(b) If $f(x)$ is differentiable at $x = 7$, then $f(x)$ is continuous at $x = 7$.

Contrapositive: If $f(x)$ is not continuous at $x = 7$, then $f(x)$ is not differentiable at $x = 7$.

Converse: If $f(x)$ is continuous at $x = 7$, then $f(x)$ is differentiable at $x = 7$.

The original statement is a theorem from calculus: differentiability implies continuity. Thus the contrapositive is equivalent to this (true) statement.

The converse, however, is false: $f(x) = |x - 7|$ is continuous everywhere, but not differentiable at $x = 7$.

(c) If n is odd, then either $n + 1$ or $n - 1$ is divisible by 6.

Contrapositive: If both $n + 1$ and $n - 1$ are not divisible by 6, then n is even.

Converse: If $n + 1$ is divisible by 6 or $n - 1$ is divisible by 6, then n is odd.

The original statement (hence its contrapositive) is false: For example, $n = 9$ is odd, but neither 8 nor 10 are divisible by 6.

The converse is true: Since if $n + 1$ or $n - 1$ is divisible by 6, then that number is divisible by 2; this implies n is odd.

2. Prove by contradiction that if a, b are real numbers with $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Suppose a, b are reals and $a \cdot b = 0$. Then assume for a contradiction that neither a nor b is equal to 0. Then both have multiplicative inverses: $aa^{-1} = bb^{-1} = 1$. But then

$$0 = 0 \cdot b^{-1}a^{-1} = (a \cdot b)b^{-1}a^{-1} = a(bb^{-1})a^{-1} = aa^{-1} = 1.$$

We have $0 = 1$. But this is a contradiction; so we must have had that a or b was equal to 0.

3. Show for all positive integers n that 4 divides $5^n + 7$.

We prove this by induction. In the base case, $n = 1$; and $5^n + 7 = 5 + 7 = 12 = 4 \cdot 3$, so we have the claim in this case.

Suppose inductively that $4|(5^k+7)$ for some integer $k \geq 1$. By definition of divides, $5^k+7=4\cdot a$ for some integer a. Now

$$\begin{aligned}
5^{k+1}+7 &= 5\cdot 5^k+7 \\
&= 5\cdot 5^k+35-28 \\
&= 5\cdot(5^k+7)-28 \\
&= 5\cdot 4\cdot a-28 \text{ (by inductive hypothesis)} \\
&= 4\cdot(5a-7).
\end{aligned}$$

Since $5a-7$ is an integer, we've shown $4|(5^{k+1}+7)$. We conclude by the principle of induction that $4|(5^n+7)$ for all positive integers n.

4. Prove that for all positive integers n,

$$\sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}.$$

We prove this identity by induction on n.

In the base case, $n=1$. Then

$$\sum_{i=1}^{1} i^2 = 1 = \frac{2+3+1}{6}$$

so the identity holds.

For the inductive step, assume that we have

$$\sum_{i=1}^{k} i^2 = \frac{2k^3+3k^2+k}{6}.$$

for some integer $k \geq 1$. We need to show the same identity holds when we replace "k" by "$k+1$" throughout. We have

$$\begin{aligned}
\textstyle\sum_{i=1}^{k+1} i^2 &= \left(\textstyle\sum_{i=1}^{k} i^2\right)+(k+1)^2 && \text{(by definition of this sum)} \\
&= \frac{2k^3+3k^2+k}{6}+(k+1)^2 && \text{(by inductive hypothesis)} \\
&= \frac{2k^3+3k^2+k}{6}+\frac{6k^2+12k+6}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6} \\
&= \frac{2(k+1)^3+3(k+1)^2+(k+1)}{6} && \text{(by arithmetic).}
\end{aligned}$$

Thus the identity for $n=k$ implies the identity for $n=k+1$. By induction, we have proved the identity holds for all positive integers n.

5. Let k,n be positive integers. Prove that if k is the least integer divisor of n such that $k>1$, then k is prime.

Fix integers $k,n>0$, and suppose that k is the least divisor of n greater than 1. Suppose for a contradiction that k is not prime; then $k=a\cdot b$ for some positive integers a,b, neither of which is 1 or k. But then $1<a<k$; and by homework 1, since $a|k$ and $k|n$ we have $a|n$. This contradicts that k was the least divisor of n, which completes the proof.

6. Show that an integer $n > 1$ is prime if and only if n has no prime divisors p with $p^2 \leq n$.

We need to show both directions of this implication. The forward direction is easy: Suppose $n > 1$ is prime. Then n has no divisors besides 1 and n; and since $n > 1$, we have $n^2 > n$. In particular, n has no prime divisors p satisfying $p^2 \leq n$: Its only prime divisor is itself, and $n^2 > n$.

For the reverse implication, we prove the contrapositive. Suppose therefore that $n > 1$ is not prime. Let $p > 1$ be least such that p divides n; by assumption that n is composite, $1 < p < n$. By the previous problem, p is prime.

Let k be such that $pk = n$; then $1 < p \leq k < n$, by minimality of p. Now

$$p^2 = p \cdot p \leq p \cdot k = n.$$

We therefore have shown the existence of a prime p such that $p|n$ and $p^2 \leq n$. This completes the proof.