Math 215 - Homework 3 Solutions

1. Give a definition (in set builder notation) of the set of natural numbers that can be written as the sum of three squares of natural numbers.

A number x can be written as a sum of three naturals if and only if there are a, b, c all in $\mathbb{N}$ such that $a^2 + b^2 + c^2 = x$. Thus, this set is equal to

$$\{x \in \mathbb{N} \mid \exists a \in \mathbb{N}\ \exists b \in \mathbb{N}\ \exists c \in \mathbb{N}\ a^2 + b^2 + c^2 = x\}.$$

Using so called "constructive" notation, this could just as well be defined by

$$\{a^2 + b^2 + c^2 \mid a, b, c \in \mathbb{N}\}.$$

2. Show $(A - B) \cap (A - C) = A - (B \cup C)$.

Proof. We show both inclusions $\subseteq$ and $\supseteq$. For "$\subseteq$", suppose $x \in (A - B) \cap (A - C)$. Then $x \in A$, $x \notin B$, and $x \notin C$. That is, x is in A but not in B or in C; $x \in A$ and $x \notin B \cup C$. Thus $x \in A - (B \cup C)$. This shows $(A - B) \cap (A - C) \subseteq A - (B \cup C)$.

For "$\supseteq$", suppose $x \in A - (B \cup C)$. That is, x is in A and not in B or C. So $x \in A$, $x \notin B$, and $x \notin C$. Thus $x \in A - B$ and in $x \in A - C$; i.e. $x \in (A - B) \cap (A - C)$. This shows $A - (B \cup C) \subseteq (A - B) \cap (A - C)$.

Taken together, these inclusions show the two sets are equal. $\square$

3. Show $A - B = \varnothing$ if and only if $A \subseteq B$.

Proof. For left-to-right implication: Suppose $A - B = \emptyset$. Then there is no $x \in A - B$; that is, for every $x \in A$, it is not the case that $x \notin B$. That is: For every $x \in A$, we have $x \in B$. Thus $A \subseteq B$.

For right-to-left implication: we show the contrapositive. Suppose then that $A - B \neq \varnothing$. That is, there is some $x \in A - B$, so that $x \in A$ and $x \notin B$. But then $A \not\subseteq B$. We have shown if $A - B \neq \varnothing$ then $A \not\subseteq B$; equivalently, if $A \subseteq B$ then $A - B = \varnothing$. This completes the proof. $\square$

4. List the elements of the set $\mathcal{P}(\{1, \{1, 2\}\})$.

$A = \{1, \{1, 2\}\}$ is a 2-element set, whose only elements are 1 and $\{1, 2\}$. The power set will have $2^2 = 4$ elements. Namely:

$$\mathcal{P}(A) = \{\varnothing, \{1\}, \{\{1, 2\}\}, \{1, \{1, 2\}\}\}.$$

If it helps, think of A as $\{a, b\}$, with $a = 1$ and $b = \{1, 2\}$; so $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, just as we wrote above.