Math 215 - Homework 4 Solutions

1. For each of the following sets A_i, give three examples of elements A_i, and three examples of elements of A_i^c.

(a) $A_1 = \{a \in \mathbb{Z} \mid \sqrt{a} \in \mathbb{Z}\}$

(b) $A_2 = \{p^q \mid p, q \in \mathbb{N} \text{ and } p, q \text{ are primes}\}$

(c) $A_3 = \{x \in \mathbb{R} \mid \text{for some } n \in \mathbb{Z},\ 3n \leq x < 3n + 2\}$

(d) $A_4 = \{\langle x, y\rangle \in \mathbb{N} \times \mathbb{R} \mid e^y = 2x\}$

A_1 is the set of squares of integers. Examples: 1,4,9,16,25,36, etc. Any number that isn't a perfect square is in the complement, e.g.: $-1, 5, 12, 15$, etc.

A_2 contains an integer n if and only if it *can be written* as a power p^q with p, q both prime. So $2^3 = 8, 2^5 = 32, 3^3 = 27$ ahd $5^3 = 125$ are all in A_2. But $6, 7, 12$, and $2^4 = 16$ are not, since these are not prime-exponent powers of primes.

Note that for any integer n, the half-open interval $[3n, 3n + 2)$ is a subset of A_3, examples: $[-6, -4)$,$[0, 2)$,$[3, 5)$. So we can pick from these sets: $-5, 0, 1, 3, \pi, 4.999, 27, 101.5$ are all in A_3. The complement A_3^c is the union of sets of the form $[3n - 1, 3n)$; examples of subsets of the complement are $[-4, -3)$,$[2, 3)$, $[98, 99)$, and so on. So e.g. $-4, -3.6, 2, 98.999$ are all in the complement.

A_4 consists of pairs of the form $\langle x, y$ with $y = \ln 2x$, where n *is a natural number*; note we must have $n > 0$. So we have $\langle 1, \ln 2\rangle$, $\langle 2, \ln 4\rangle$, $\langle 3, \ln 6\rangle$ are typical elements of A_4. But things like $\langle 1/2, 1\rangle$, $\langle 2, 12\rangle$ and $\langle 5, 5\rangle$ are in the complement.

2. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be defined by $f(x, y) = x - y$.

(a) Give a table of values for $f|\{0, 1, 2, 3\}^2$.

(b) Is f injective? Surjective? Prove your answers.

(a) Table:

x \ y	0	1	2	3
0	0	-1	-2	-3
1	1	0	-1	-2
2	2	1	0	-1
3	3	2	1	0

(b) It's clear from the table that f is not injective; for one example, $f(\langle 1, 1\rangle) = 0 = f(\langle 2, 2\rangle)$.

f is surjective, however. Suppose n is an integer. If $n \geq 0$, then $\langle n, 0\rangle \in \mathbb{N} \times \mathbb{N}$, and we have $f(n, 0) = n - 0 = n$; on the other hand, if $n < 0$, then we have $\langle 0, -n\rangle \in \mathbb{N}$, and $f(\langle 0, -n\rangle) = 0 - (-n) = n$.

3. Let A, B be sets such that $|A| = 7$, $|B| = 29$, and $|A \cup B| = 32$. Determine the sizes of each of the following sets:

(a) $A \cap B$

(b) $B - A$

(c) $A \times B$

(d) $(A \times A) - \{\langle a, b\rangle \in A^2 \mid a = b\}$

(a) Since $|A \cup B| = |A| + |B| - |A \cap B|$, we have $|A \cap B| = (7 + 29) - 32 = 4$.
(b) $|B - A| = |B| - |A \cap B| = 29 - 4 = 25$.
(c) $|A \times B| = |A| \cdot |B| = 7 \cdot 29 = 203$.
(d) Since $\{\langle a, b\rangle \in A^2 \mid a = b\} = \{\langle a, a\rangle \mid a \in A\}$, this set has the same size as A. So $|(A \times A) - \{\langle a, b\rangle \in A^2 \mid a = b\}| = 49 - 7 = 42$.

4. Let $f : A \to B$ be a function. Show that f is injective if and only if for all sets $X, Y \subseteq A$,

$$\mathrm{Im}(f|(X \cap Y)) = \mathrm{Im}(f|X) \cap \mathrm{Im}(f|Y).$$

($\Rightarrow$) Suppose f is injective. We need to show the displayed equality holds. We do this by showing both inclusions. For $\subseteq$: Suppose $y \in \mathsf{Im}(f|(X \cap Y)$. So $y = f(x)$ for some $x \in X \cap Y$. Since $x \in X$, $y = f(x) \in \mathsf{Im}(f|X)$; similarly $x \in Y$ implies $y = f(x) \in \mathsf{Im}(f|Y)$. So $y \in \mathsf{Im}(f|X) \cap \mathsf{Im}(f|Y)$, which proves $\subseteq$ in the above equality (note this inclusion did not use injectivity of f).

Now for the reverse inclusion, $\supseteq$: suppose $y \in \mathsf{Im}(f|X) \cap \mathsf{Im}(f|Y)$; so $y = f(x_1)$ for some $x_1 \in X$, and $y = f(x_2)$ for some $x_2 \in Y$. That is, $f(x_1) = f(x_2)$, so by injectivity of f, $x_1 = x_2$. Thus $x_1 \in X \cap Y$, and we have $y = f(x_1) \in \mathsf{Im}(f|(X \cap Y))$.

($\Leftarrow$) Suppose now that $\mathsf{Im}(f|(X \cap Y)) = \mathsf{Im}(f|X) \cap \mathsf{Im}(f|Y)$ whenever X, Y are subsets of A. We wish to prove f is injective. For this, fix $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Note that then if $X = \{a_1\}$ and $Y = \{a_2\}$, then $X \cap Y = \emptyset$; it follows that $\mathsf{Im}(f|(X \cap Y)) = \emptyset$. Now $\mathsf{Im}(f|X) = \{f(a_1)\}$, and $\mathsf{Im}(f|Y) = \{f(a_2)\}$. By assumption, these sets must be disjoint; and this is the case precisely when $f(a_1) \neq f(a_2)$. This proves f is injective.

5. Recall $\mathbb{R}^{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\} = [0, \infty)$. For each of the following, give an example of a function $f_i : \mathbb{R}^{\geq 0} \to \mathbb{R}$ with the listed properties.

(a) f_1 is injective, but not surjective.

(b) f_2 is surjective, but not injective.

(c) f_3 is bijective.

(a) Just take $f_1(x) = x$; clearly this is injective, but takes no negative values, so is not onto $\mathbb{R}$.
(b) We could take, for example:

$$f_2(x) = \begin{cases} \log(x) & \text{if } x > 0; \\ 256 & \text{if } x = 0. \end{cases}$$

Note just taking $f_2 = \log$ would not be enough, since this wouldn't be a function on all of $[0, \infty)$; also, it didn't matter what value we picked for $f_2(x)$, since no matter what we choose f_2 will fail to be surjective (since $\log$ "used up" all reals: Its range is already all of $\mathbb{R}$).

(c) With this part we must be more creative; indeed, it can be shown that any bijection as in this problem must fail to be continuous at infinitely many points, so a piecewise definition of some kind is necessary.

Here is one example:

$$f_3(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - n & \text{if } 2n < x \le 2n+1,\ n \in \mathbb{N} \\ n - x & \text{if } 2n-1 < x \le 2n,\ n \in \mathbb{N} \text{ and } n > 0 \end{cases}$$

Note this is well-defined: Given any $x \in (0, \infty)$, there is a unique natural k such that $k < x \le k+1$; and this k is either even ($k = 2n$ some n) or or odd ($k = 2n-1$), and not both. So precisely one of the three cases will be used to define $f_3(x)$.

It may be instructive to sketch a graph of f_3, and I **strongly encourage you to do so!**

Let's check that this is one-to-one and onto. First notice that $x > 0$ implies $|f_3(x)| = |x - n|$ where $n < x/2$; in particular $|f_3(x)| > |x/2| > 0$, so $f_3(x) \neq 0$ whenever $x \neq 0$.

Now notice that if $x \in (2n, 2n+1]$, that

$$n = 2n - n < x - n \le (2n+1) - n = n+1;$$

so $\mathrm{Im}(f_3|(2n, 2n+1]) \subseteq (n, n+1]$. In fact we have an equality here, since $f_3(2n+1) = n+1$ and $f_3|(2n, 2n+1]$ has slope 1.

Similarly we can check $\mathrm{Im}(f_3|(2n-1, 2n]) = [-n, -n+1)$ as follows:

$$\begin{aligned} &2n - 1 < x \le 2n \\ \Rightarrow\ &n - 1 < x - n \le n \\ \Rightarrow\ &1 - n > n - x \ge -n, \end{aligned}$$

which gives $\subseteq$; and again since $f_3(2n) = n - 2n = -n$ and $f_3|(2n-1, 2n]$ has slope -1, the range of this restriction is all of $[-n, -n+1)$.

Since $\mathbb{R}$ is the disjoint union of the sets $\{0\}$, $(n, n+1]$ for $n \in \mathbb{N}$, and $[-n, -n+1)$ for $n \in \mathbb{N} - \{n\}$, we have that f_3 is onto $\mathbb{R}$. Note also that since this family is pairwise disjoint, we have that $f_3(x) \neq f_3(y)$ whenever $x \in (m, m+1]$, $y \in (n, n+1]$ for distinct naturals m, n; and if x, y are in the same piece $(n, n+1]$, clearly $x \neq y$ implies $f_3(x) \neq f_3(y)$.

This shows that $f_3(x)$ is onto and one-to-one, hence bijective.

6. Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by

$$g(x, y) = (x + y + 1)(x + y) + 2y.$$

(a) Give a table of values for $g|\{0, 1, 2, 3\}^2$.

(b) Is g injective? Surjective? Prove your answers.

textsf(a) Table:

x \ y	0	1	2	3
0	0	4	10	18
1	2	8	16	26
2	6	14	24	36
3	12	22	34	48

(b) From the table, we can guess that g is not surjective, but appears to be injective—indeed it appears to be listing just the even numbers, in increasing order going across up and right diagonals. (You could expand this table to confirm this suspicion.)

Proving g is not surjective is somewhat straightforward; for any choice of $x + y$, precisely one of $(x + y + 1), (x + y)$ is even, and since even $\cdot$ odd $=$ even, we have that $(x + y + 1)(x + y) = 2k$ for some k; thus $g(x, y) = 2k + 2y = 2(k + y)$ is even. Since all outputs of g are even, it cannot be onto $\mathbb{N}$.

Showing g is injective is more challenging. We need to show that if $\langle x, y\rangle, \langle x', y'\rangle$ are distinct pairs of naturals, then $g(x, y) \neq g(x', y')$. Let us denote $n = x + y$ and $n' = x' + y'$. So

$$g(x, y) = (n + 1)n + 2y, \quad g(x', y') = (n' + 1)n' + 2y'.$$

Note that if $n = n'$, then we must have $y \neq y'$, which implies $g(x', y') = (n+1)n+2y' \neq (n+1)n+2y = g(x, y)$. So we may assume that $n < n'$. We claim that $g(x, y) < g(x', y')$ (note this corresponds to all outputs of g on the nth up-and-right diagonal being less than those on the n'th).

Now since $n < n'$ and these are naturals, we have $n + 1 \leq n'$. So we get

$$(n + 1)n = n^2 + 1 = (n^2 + 3n + 2) - 3n - 1 = (n + 2)(n + 1) - 3n - 1 \leq (n' + 1)n' - 3n - 1.$$

Note also that $n = x + y$ and $x, y \in \mathbb{N}$ together imply $y \leq n$. Putting this all together:

$$\begin{aligned}
g(x, y) &= (n + 1)n + 2y \\
&\leq (n + 1)n + 2n \\
&\leq [(n' + 1)n' - 3n - 1] + 2n \\
&= (n' + 1)n' - n - 1 \\
&< (n' + 1)n' \\
&\leq (n' + 1)n' + 2y' \\
&= g(x', y').
\end{aligned}$$

We got the strict inequality because $n + 1 > 0$ for all naturals n. So we have $g(x, y) < g(x', y')$ whenever $x + y < x' + y'$; together with the case $n = n'$ covered above, this shows that g is an injection.