Math 215 - Homework 5

Due ~~Friday, October 29~~ Monday, October 28

1. Suppose $f : A \to B$ is a bijection, and that A is a proper subset of B. Show A is infinite.

2. Suppose n dogs and $k \leq n$ cats are to sit at a circular table (with $n + k$ chairs around it) in such a way that no two cats are sitting next to each other. How many seating arrangements are possible? Carefully explain your answer.

3. Suppose $A \subseteq \mathbb{N}_n = \{1, 2, \ldots, n\}$, and that $k = |A|$ satisfies

$$\frac{k(k-1)}{2} > n - 1.$$

Show there is some "distance" between elements of A which occurs at least twice; that is, there are $a_1 < a_2$ and $b_1 < b_2$, all in A, with $a_1 \neq b_1$, such that

$$a_2 - a_1 = b_2 - b_1$$

4. A sequence $s : \mathbb{N}_k \to \mathbb{N}$ is **decreasing** if $i < j \leq k$ implies $s(i) > s(j)$. There is a unique decreasing sequence whose first entry is 0 (namely, $\langle 0 \rangle$) and two with first entry 1 (these are $\langle 1, 0 \rangle$ and $\langle 1 \rangle$.)

How many decreasing sequences $s : \mathbb{N}_k \to \mathbb{N}$ ($k \in \mathbb{N}$) with first entry n are there? Prove your answer.

5. Consider finite sequences whose entries are *subsets* of $\mathbb{N}$, $s : \mathbb{N}_k \to \mathcal{P}(\mathbb{N})$. We now say s is **decreasing** (inclusionwise) if $i < j \leq k$ implies $s(i) \supsetneq s(j)$, that is, $s(j)$ is a *proper* subset of $s(i)$. Let $F(n)$ for $n \in \mathbb{N}$ be the number of decreasing sequences s of subsets of $\mathbb{N}$ such that $s(1) = \mathbb{N}_n$.

(a) Show $F(0) = 1$ and $F(1) = 2$.

(b) What are $F(2), F(3)$ and $F(4)$?

(c) Find an inductive definition for F. That is, find an expression for $F(n+1)$ in terms of $F(0), F(1), \ldots, F(n)$.