Math 215 - Homework 5 Solutions

1. Suppose $f : A \to B$ is a bijection, and that A is a proper subset of B. Show A is infinite.

Suppose as in the hypotheses that $A \subsetneq B$ and that $f : A \to B$ is a bijection. Suppose, for a contradiction, that A is finite. Since A and B are in bijective correspondence, B is finite as well.

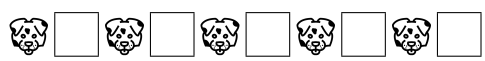
Since A is a proper subset of B, we have $B - A$ is nonempty. By the addition principle, $|B| = |A| + |B - A|$, and $|B - A| \geq 1$ since A is a *proper* subset; in particular, $|A| < |B|$.

Now $f^{-1} : B \to A$ is an injection. By the pigeonhole principle, $|B| \leq |A|$. But this contradicts the conclusion of the previous paragraph. We must have that A is infinite.

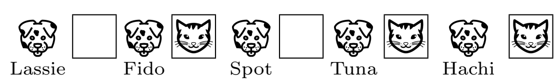
2. Suppose n dogs and $k \leq n$ cats are to sit at a circular table (with $n + k$ chairs around it) in such a way that no two cats are sitting next to each other. How many seating arrangements are possible? Carefully explain your answer.

For illustration, let's consider 5 dogs and 3 cats. Each seating arrangement is uniquely determined by some ordering of the animals, so let's concentrate on this.

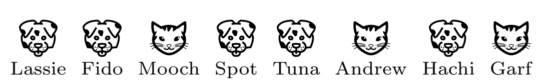
If we line up the dogs in a row, we have one space between each pair (plus one at the end, for when they sit down) where we can put a cat, for 5 spaces total.



Note there are $5!$ ways to line up the dogs. For each of these, we have $\binom{5}{3}$ ways to pick spaces where the cats will sit, for example



Once these positions are chosen, we may forget about the unused spaces, and order the cats, which we have $3!$ ways to do; for example,



So we have $5!\binom{5}{3}3! = 60$ ways to line up the animals in this manner. For general k and n, we get $n!k!\binom{n}{k}$ ways to order the animals.

Now we also have a choice as to what our final answer should actually be counting. First, we can imagine the seats are numbered, so we have a one-to-one correspondence between seating arrangements and orderings, and our answer is the same: $n!k!\binom{n}{k}$ (this is how I interpreted the problem when I was asked about this in class).

Alternatively, we may want to regard two seating arrangements as identical if they are obtained from one another by a rotation (so all we care about is the relative positions of the animals). Since there are $n + k$ rotations of the table, we have overcounted the number of arrangements by a factor of $n + k$, and we obtain our final answer by dividing to correct for this: $n!k!\binom{n}{k}/(n + k)$.

3. Suppose $A \subseteq \mathbb{N}_n = \{1, 2, \ldots, n\}$, and that $k = |A|$ satisfies

$$\frac{k(k-1)}{2} > n - 1.$$

Show there is some "distance" between elements of A which occurs at least twice; that is, there are $a_1 < a_2$ and $b_1 < b_2$, all in A, with $a_1 \neq b_1$, such that

$$a_2 - a_1 = b_2 - b_1$$

Recall $\mathcal{P}_2(A)$ is the set of 2-element subsets of A. Define a function $f : \mathcal{P}_2(A) \to \mathbb{N}_{n-1}$ by: $f(\{x, y\}) = |x - y|$. Note this f is well-defined: If we have $\{x, y\} \in \mathcal{P}_2(A)$, say with $x < y$, then since $1 \leq x < y \leq n$, we have that $1 \leq y - x \leq n - 1$, so $f(\{x, y\}) \in \mathbb{N}_{n-1}$.

Now, $|A| = k$, so $|\mathcal{P}_2(A)| = \binom{k}{2} = k(k-1)/2$; and by assumption, $k(k-1)/2 > n - 1 = |\mathbb{N}_{n-1}|$.

We thus have a map $f : \mathcal{P}_2(A) \to \mathbb{N}_{n-1}$, and $|\mathcal{P}_2(A)| > |\mathbb{N}_{n-1}|$. By the pigeonhole principle, f is not injective; there must exist distinct subsets $\{a_1, a_2\}, \{b_1, b_2\}$ of A so that $d := f(\{a_1, a_2\}) = f(\{b_1, b_2\})$; we may assume $a_1 < a_2$ and $b_1 < b_2$. Note then $a_1 \neq b_1$, as otherwise $a_2 = a_1 + d = b_1 + d = b_2$, which would contradict these sets being distinct. By our definition of f, we have

$$a_2 - a_1 = f(\{a_1, a_2\}) = f(\{b_1, b_2\}) = b_2 - b_1,$$

as desired.

4. A sequence $s : \mathbb{N}_k \to \mathbb{N}$ is **decreasing** if $i < j \leq k$ implies $s(i) > s(j)$. There is a unique decreasing sequence whose first entry is 0 (namely, $\langle 0 \rangle$) and two with first entry 1 (these are $\langle 1, 0 \rangle$ and $\langle 1 \rangle$.)

How many decreasing sequences $s : \mathbb{N}_k \to \mathbb{N}$ ($k \in \mathbb{N}$) with first entry n are there? Prove your answer.

For $n \in \mathbb{N}$, let D_n be the set of decreasing sequences whose first entry is n. We claim $|D_n| = 2^n$.

There are a few ways to see this. One is inductive: Clearly $|D_0| = 1$, and a sequence of length n is either the one-element sequence $\langle n \rangle$, or is obtained by adjoining n to the beginning of a sequence from D_k for some $k < \mathbb{N}$. So D_n is in one-to-one correspondence with $\{\langle n \rangle\} \cup \bigcup_{i=0}^{n-1} D_i$. Thus by the addition principle,

$$|D_n| = 1 + |D_0| + |D_1| + \cdots + |D_{n-1}| = 1 + 2^0 + 2^1 + \cdots + 2^{n-1} = 1 + \sum_{i=0}^{n-1} 2^i = 2^n.$$

Note here the second equality is by inductive hypothesis.

We could also more simply note that a *decreasing* sequence whose first entry is n is uniquely determined by its *set* of entries less than n: We simply list them in decreasing order. Thus we have a bijection between D_n and $\mathcal{P}(\{0, 1, \ldots, n-1\})$, and the power set of an n-element set has cardinality 2^n.

5. Consider finite sequences whose entries are *subsets* of $\mathbb{N}$, $s : \mathbb{N}_k \to \mathcal{P}(\mathbb{N})$. We now say s is **decreasing** (inclusionwise) if $i < j \leq k$ implies $s(i) \supsetneq s(j)$, that is, $s(j)$ is a *proper* subset of $s(i)$. Let $F(n)$ for $n \in \mathbb{N}$ be the number of decreasing sequences s of subsets of $\mathbb{N}$ such that $s(1) = \mathbb{N}_n$.

(a) Show $F(0) = 1$ and $F(1) = 2$.

(b) What are $F(2), F(3)$ and $F(4)$?

(c) Find an inductive definition for F. That is, find an expression for $F(n+1)$ in terms of $F(0), F(1), \ldots, F(n)$.

For this problem, let's denote the set of decreasing sequences with first entry $\mathbb{N}_n$ by S_n; so $F(n) = |S_n|$ for all n.

(a) There is a unique set of size 0, namely $\varnothing$, and this has no *proper* subsets. So $S_0 = \{\langle\varnothing\rangle\}$. By similar reasoning, the only decreasing sequences with first entry $\mathbb{N}_1$ are $\langle\{1\}\rangle$ and $\langle\{1\},\varnothing\rangle$.

(b) and (c). The situation is now more complicated, but as in problem 4, notice that if a sequence s has first entry $\mathbb{N}_n$, then either it has length one, or it is obtained by adjoining a sequence whose first entry has size k with $k < n$; however, there are $\binom{n}{k}$ such possible sets to choose from for each k.

Suppose $A \in \mathcal{P}_k(\mathbb{N})$. The number of decreasing sequences with first entry A is the same as the cardinality of S_k (what is the bijection?), and this is just $F(k)$, no matter which set A is.

$$|S_2| = 1 + \binom{2}{0}|S_0| + \binom{2}{1}|S_1| = 1 + 1 + 4 = 6;$$

and

$$|S_3| = 1 + \binom{3}{0}|S_0| + \binom{3}{1}|S_1| + \binom{3}{2}|S_2| = 1 + 1 + 3\cdot 2 + 3 \cdot 6 = 26;$$

and

$$|S_4| = 1 + \binom{4}{0}|S_0| + \binom{4}{1}|S_1| + \binom{4}{2}|S_2| + \binom{4}{3}|S_3| = 1 + 1 + 4\cdot 2 + 6\cdot 6 + 4\cdot 26 = 150.$$

As the reasoning above shows, we select a decreasing sequence whose first entry is $\mathbb{N}_{n+1}$ as follows: first (1) either (1a) don't extend the sequence at all, or (1b) choose some natural $k \le n$ (so step (1) gives us $n+2$ summands in all); then (2) if we chose a natural k, we choose a subset A of $\mathbb{N}_{n+1}$ of size k (for which we have $\binom{n+1}{k}$ choices); and finally (3) choosing a decreasing sequence with first entry A, for which we have $F(k)$ choices. (Why does this count each sequence exactly once?)

So we have the inductive definition

$$F(n+1) = 1 + \binom{n+1}{0}F(0) + \binom{n+1}{1}F(1) + \binom{n+1}{2}F(2) + \cdots + \binom{n+1}{n}F(n),$$

or more compactly,

$$F(n+1) = 1 + \sum_{i=0}^{n}\binom{n+1}{i}F(i).$$