Math 215 - Homework 6

Due Friday, November 16

1. For each of the following relations R_i, determine whether R_i is an equivalence relation on $\mathbb{R}$. Prove your answer.

(a) $x\ R_1\ y$ iff $y - x \in \mathbb{Z}$.

(b) $x\ R_2\ y$ iff $y - x \geq 0$.

(c) $x\ R_3\ y$ iff $|y - x| < 1$.

2. Recall the floor function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by $\lfloor x \rfloor =$ the greatest integer n such that $n \leq x$.

For each of the following equivalence relations R_i on $\mathbb{R}$, determine whether the definition $[x]_{R_i} \oplus [y]_{R_i} = [x + y]_{R_i}$ is a well-defined binary operation on the R_i-equivalence classes. Prove your answers.

(a) $x\ R_4\ y$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

(b) $x\ R_5\ y$ iff $x - \lfloor x \rfloor = y - \lfloor y \rfloor$.

(c) $x\ R_6\ y$ iff $y - x \in \mathbb{Q}$.

3. Recall for all reals x that the interval $[x, x + 1)$ contains exactly one integer a.

Use this fact and properties of the order $<$ on $\mathbb{R}$ to show the following.

(a) For all reals x, there is an integer $n > x$.

(b) For all positive reals ε, there is an integer $n > 0$ with $1/n < \varepsilon$.

(c) For all pairs of reals $x < y$, there is a rational number p with $x < p < y$.

Let A be a set. For the next problems, we let $A^{\mathbb{N}}$ denote the set of infinite sequences in A,

$$A^{\mathbb{N}} := \mathrm{Fun}(\mathbb{N}, A),$$

and $A^{<\mathbb{N}}$ denotes the set of finite sequences in A,

$$A^{<\mathbb{N}} = \bigcup_{k \in \mathbb{N}} \mathrm{Fun}(\mathbb{N}_k, A).$$

4. Show the relation E on $\mathbb{N}^{\mathbb{N}}$ defined by

$$\alpha\ E\ \beta \text{ iff } \{n \in \mathbb{N} \mid \alpha(n) \neq \beta(n)\} \text{ is finite},$$

for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, is an equivalence relation.

5. Show $\mathbb{N}^{<\mathbb{N}}$ is countable.

6. Use the Cantor-Shröder-Bernstein Theorem to show $|\mathbb{R}| = |\{0, 1\}^{\mathbb{N}}|$.