Math 215 - Homework 6 Solutions

1. For each of the following relations R_i, determine whether R_i is an equivalence relation on $\mathbb{R}$. Prove your answer.

(a) $x\, R_1\, y$ iff $y - x \in \mathbb{Z}$.

(b) $x\, R_2\, y$ iff $y - x \geq 0$.

(c) $x\, R_3\, y$ iff $|y - x| < 1$.

(a) This is an equivalence relation. We check each of the three properties:

- Reflexivity: For all $x \in \mathbb{Z}$, $x - x = 0 \in \mathbb{Z}$. So $x\, R_1\, x$ for all $x \in \mathbb{R}$.
- Symmetry: Suppose $x\, R_1\, y$. So $y - x \in \mathbb{Z}$, and $x - y = -(y - x) \in \mathbb{Z}$ as well. So $y\, R_1\, z$.
- Transitivity: Suppose $x, y, z \in \mathbb{R}$ are such that $x\, R_1\, y$ and $y\, R_1\, z$. Then $y - x = k_1 \in \mathbb{Z}$, and $z - y = k_2 \in \mathbb{Z}$. So $z - x = z - y + y - x = k_2 + k_1 \in \mathbb{Z}$, since $\mathbb{Z}$ is closed under addition. We have $x\, R_1\, z$, so that R_1 is transitive.

(b) R_2 is not an equivalence relation. It is not symmetric, for example: $2 - 1 = 1 \geq 0$, so $1\, R_2\, 2$; but $1 - 2 = -1 < 0$, so that $\neg(2\, R_2\, 1)$.

(c) R_3 is not an equivalence relation. It is reflexive and symmetric, but it is not transitive: For example, $0\, R_3\, 0.8$ and $0.8\, R_3\, 1.5$, but $\neg(0\, R_3\, 1.5)$.

2. Recall the floor function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by $\lfloor x \rfloor =$ the greatest integer n such that $n \leq x$.

For each of the following equivalence relations R_i on $\mathbb{R}$, determine whether the definition $[x]_{R_i} \oplus [y]_{R_i} = [x + y]_{R_i}$ is a well-defined binary operation on the R_i-equivalence classes. Prove your answers.

(a) $x\, R_4\, y$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

(b) $x\, R_5\, y$ iff $x - \lfloor x \rfloor = y - \lfloor y \rfloor$.

(c) $x\, R_6\, y$ iff $y - x \in \mathbb{Q}$.

(a) $\oplus$ is not well-defined on the R_4-equivalence classes. For example, $0\, R_4\, 0.6$ and $2\, R_4\, 2.6$. But $0 + 2 = 2$ and $0.6 + 2.6 = 3.2$, and $\neg(2\, R_4\, 3.2)$.

(b) This is well-defined. Proof: Note that this is the same relation as R_1 in problem 1(a). To see this, note that $x - \lfloor x \rfloor = y - \lfloor x \rfloor$ implies $x - y = \lfloor x \rfloor + \lfloor y \rfloor$, and the right hand side is a sum of integers. Conversely, $\lfloor x + n \rfloor = n + \lfloor x \rfloor$ for all integers n, so $x + n - \lfloor x + n \rfloor = x + n - n - \lfloor x \rfloor = x - \lfloor x \rfloor$. So $x\, R_5\, (x + n)$ for all integers n. This shows $x\, R_5\, y$ if and only if $x - y$ is an integer.

Suppose we have x_1, x_2, y_1, y_2 such that $[x_1]_{R_5} = [x_2]_{R_5}$ and $[y_1]_{R_5} = [y_2]_{R_5}$. Note that $x_1\, R_5\, x_2$ says precisely that the non-integer parts of x_1, x_2 are the same. In particular, $x_1 R_5 x_2$ means $x_1 = n_1 + a$ and $x_2 = n_2 + a$, for some integers $n_1, n_2 \in \mathbb{Z}$ and a real a, $0 \leq a < 1$ (where $a = x_i - \lfloor x_i \rfloor$, $i \in \{1, 2\}$). Similarly, $y_1 = m_1 + b$ and $y_2 = m_2 + b$ with $m_1, m_2 \in \mathbb{Z}$ and $0 \leq b < 1$.

So now $x_1+y_1 = (n_1+a)+(n_2+b) = (n_1+n_2)+(a+b)$. And $x_2+y_2 = (m_1+m_2)+(a+b)$. And since $(n_1+n_2)+(a+b)$ and $(m_1+m_2)+(a+b)$ differ by an integer, we have $[x_1+y_1]_{R_5} = [x_2+y_2]_{R_5}$, by the remarks above. This shows $\oplus$ is well-defined.

(c) This is well-defined. Suppose $x_1\ R_6\ x_2$ and $y_1\ R_6\ y_2$. Then set $p = x_2 - x_1 \in \mathbb{Q}$ and $q = y_2 - y_1 \in \mathbb{Q}$. We have $x_1 + y_1 = (x_2 - p) + (y_2 - q) = (x_2 + y_2) - (p + q)$; since these differ by a rational, we have $(x_1 + y_1)\ R_6\ (x_2 + y_2)$, which shows $\oplus$ is well-defined on the R_6-equivalence classes.

3. Recall for all reals x that the interval $[x, x + 1)$ contains exactly one integer a.
Use this fact and properties of the order $<$ on $\mathbb{R}$ to show the following.

(a) For all reals x, there is an integer $n > x$.

(b) For all positive reals ε, there is an integer $n > 0$ with $1/n < \varepsilon$.

(c) For all pairs of reals $x < y$, there is a rational number p with $x < p < y$.

(a) Let x be a real number. Then $[x + 1, x + 2)$ contains some integer n, so that $x < x + 1 \leq n$ with n an integer as needed.

(b) Let $\varepsilon > 0$. Then also $0 < \frac{1}{\varepsilon}$, and by part (a), there is an integer $n \geq 1$ with $n > \frac{1}{\varepsilon}$. Multiplying both sides by ε/n doesn't reverse the inequality (since ε/n is positive) so we obtain $\varepsilon > \frac{1}{n}$ as needed.

(c) Fix $x < y$. Then $y - x > 0$. By part (b), there is an integer n with $\frac{1}{n} < y - x$. So multiplying by $2n$, we have $2 < 2ny - 2nx$. In particular, $2nx < 2nx + 1 < 2nx + 2 < 2ny$. By the fact we are allowed to assume, there is some integer in $[2nx + 1, 2nx + 2)$, say $2nx + 1 \leq m < 2nx + 2$ with $m \in \mathbb{Z}$. So by transitivity of $<$, $2nx < m < 2ny$. Dividing through by $2n$, $x < \frac{m}{2n} < y$. And $\frac{m}{2n} \in \mathbb{Q}$, so we have a rational strictly between x and y, which is what we needed to prove.

Let A be a set. For the next problems, we let $A^{\mathbb{N}}$ denote the set of infinite sequences in A,

$$A^{\mathbb{N}} := \mathrm{Fun}(\mathbb{N}, A),$$

and $A^{<\mathbb{N}}$ denotes the set of finite sequences in A,

$$A^{<\mathbb{N}} = \bigcup_{k\in\mathbb{N}} \mathrm{Fun}(\mathbb{N}_k, A).$$

4. Show the relation E on $\mathbb{N}^{\mathbb{N}}$ defined by

$$\alpha\, E\, \beta \text{ iff } \{n \in \mathbb{N} \mid \alpha(n) \neq \beta(n)\} \text{ is finite},$$

for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, is an equivalence relation.

Let us show E is reflexive. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$, that is, α is a function $\alpha : \mathbb{N} \to \mathbb{N}$. Then for all n, $\alpha(n) = \alpha(n)$. So the set $\{n \in \mathbb{N} \mid \alpha(n) \neq \alpha(n)\}$ is the empty set; in particular, it is finite. So $\alpha\, E\, \alpha$. So E is reflexive.

Now let's show symmetry. But this is straightforward, since

$$\{n \in \mathbb{N} \mid \alpha(n) \neq \beta(n)\} = \{n \in \mathbb{N} \mid \beta(n) \neq \alpha(n)\},$$

for any sequences $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. In particular, one set is finite iff the other is; that is, $\alpha\, E\, \beta$ iff $\beta\, E\, \alpha$.

Now, transitivity. Suppose we have $\alpha, \beta, \gamma \in \mathbb{N}^{\mathbb{N}}$, such that $\alpha\, E\, \beta$ and $\beta\, E\, \gamma$. We need to show $\alpha\, E\, \gamma$, that is, that the set

$$\{n \in \mathbb{N} \mid \alpha(n) \neq \gamma(n)\}$$

is finite. This will follow if we show that

$$\{n \in \mathbb{N} \mid \alpha(n) \neq \gamma(n)\} \subseteq \{n \in \mathbb{N} \mid \alpha(n) \neq \beta(n)\} \cup \{n \in \mathbb{N} \mid \beta(n) \neq \gamma(n)\},$$

since by assumption, the right hand side is the union of two finite sets, which is finite (and a subset of a finite set is finite).

So we need to show that if $n \in \mathbb{N}$ is such that $\alpha(n) \neq \gamma(n)$, then either $\alpha(n) \neq \beta(n)$ or $\beta(n) \neq \gamma(n)$. Suppose not: We would have $\alpha(n) = \beta(n)$ and $\beta(n) = \gamma(n)$. In which case, by transitivity of "$=$", we'd have $\alpha(n) = \gamma(n)$, contrary to assumption.

This shows E is transitive.