Math 215 - Homework 7 Solutions

1. Suppose $X \subseteq Y$ and X is uncountable. Show Y is uncountable.

Proof. Fix $X \subseteq Y$. We need to show that if X is uncountable, then Y is uncountable. We show the contrapositive. Suppose Y is not uncountable; that is, Y is countable. Then $X \subseteq Y$ implies X is countable also (a theorem from class). So X is not uncountable.

2. Suppose there is an injection $f : \mathbb{R} \to X$. Show X is uncountable.

Proof. There are a few ways to do this. One is to note that since f is an injection, the function $\bar{f} : \mathbb{R} \to \mathrm{Im}(f)$ defined by $\bar{f}(x) = f(x)$ is a bijection (since by definition of range, for all $y \in \mathrm{Im}(\mathbb{R})$ we have some $x \in \mathbb{R}$ with $\bar{f}(x) = f(x) = y$; this shows $\bar{f}$ is surjective, and it is injective since f is).

Now if X were countable we would have a bijection $g : \mathbb{N} \to X$. But the composition $\bar{f}^{-1} \circ g : \mathbb{N} \to \mathbb{R}$ is a bijection. This contradicts the fact that $\mathbb{R}$ is uncountable.

3. Suppose there is a surjection $f : \mathbb{N} \to X$. Show X is countable.

Proof. Let f, X be as described. We will be done if we can find an injection $g : X \to \mathbb{N}$, since then X is in one-to-one correspondence with $\mathrm{Im}(g)$ which, being a subset of $\mathbb{N}$, is countable. Our idea is to try to "invert" f, though of course no two-sided inverse need exist since we do not assume f is bijective.

For each $a \in X$, let $A_a = \{n \in \mathbb{N} \mid f(n) = a\}$ (this is the preimage of $\{a\}$ under f). Since f is surjective, we have for every $a \in X$ that there exists $n \in \mathbb{N}$ with $f(n) = a$; in particular, A_a is non-empty for each $a \in X$. So we can set

$$g(a) = \min A_a$$

For each $a \in X$. Then $g : X \to \mathbb{N}$ is well-defined, since the minimum of a non-empty set of naturals always exists.

We need to show that g is one-to-one. Suppose a, b are elements of X, and that $g(a) = g(b)$; say n is this value. Then $n = g(a) = \min A_a = \min A_b = g(b)$. In particular, we have that $n \in A_a \cap A_b$, so that $f(n) = a$ and $f(n) = b$. This shows $a = b$, so g is injective.

By the remarks above, we have that X is countable.

4. Show $\mathbb{N}^{<\mathbb{N}}$ is countable.

(Omitted as extra credit)

5. Use the Cantor-Shröder-Bernstein Theorem to show $|\mathbb{R}| = |\{0,1\}^{\mathbb{N}}|$.

The Cantor-Shröder-Bernstein Theorem says that if there are injections $f : A \to B$ and $g : B \to A$ (that is, $|A| \le |B|$ and $|B| \le |A|$), then there is a bijection $h : A \to B$ (that is, $|A| = |B|$). So to apply this theorem, what we need to do is to show there are injections $f : \mathbb{R} \to \{0,1\}^{\mathbb{N}}$ and $g : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$.

g is a bit easier, and there's a sort of obvious example; since a decimal expansion is a sequence integers, and we need to define $g(s)$ where $s = \langle s_n \rangle_{n \in \mathbb{N}}$, $s_n \in \{0,1\}$ for all n. We can just take $g(s)$ to be the real whose decimal expansion is the entries of s. That is,

$$g(s) = 0.s_0 s_1 s_2 s_3 \cdots = \sum_{n=1}^{\infty} \frac{s_n}{10^n}.$$

Then g is injective; indeed if n is least such that $s_n \neq t_n$ for sequences s and t, then we will have $|g(s) - g(t)| \geq \frac{1}{10^n}$.

Remark: This doesn't work if we had used *binary* expansions. For example, in binary, $0.1 = 0.0\bar{1}$, so $\langle 1, 0, 0, 0, 0... \rangle$ and $\langle 0, 1, 1, 1, 1, 1, ... \rangle$ define the same real number.

Defining the injection f is somewhat trickier, and there isn't necessarily an obvious choice, since now we must somehow encode real numbers as infinite binary sequences. One option is just to use binary expansions. In some sense this is what computers do, and the same coding scheme can be employed here (though computers, having finite memory, can really only work with rational numbers up to some finite denominator); things get somewhat tricky, though, since we have to somehow encode the power of 2 the expansion "starts at", as well as distinguish negative and positive numbers.

Instead we state a somewhat less clever binary coding. Let $f(x)$, for reals x, be a binary sequence that records the decimal expansion

$$x = \pm a_0.a_1a_2a_3a_4 \dots a_n \dots$$

(where all $a_i \geq 0$) by listing the a_i as i-length blocks of 1's, separated by zeroes, and reserving the first spot for "positive" or "negative"; say, $f(x)(0) = 1$ iff $x \geq 0$. So

$$f(x) = \langle \text{(sign of x)}, (a_0\text{-many } 1\text{'s}), 0, (a_1\text{-many } 1\text{'s}), 0, (a_2\text{-many } 1\text{'s}), \dots \rangle$$

(one can define this somewhat more "rigorously" using finite sums, but it's probably not worth it).

So for example,

$$f(\pi) = \langle 1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, \dots \rangle$$

reflecting the fact that $\pi = +3.14159\dots$. And

$$f(-2.003) = \langle 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \rangle.$$

To ensure f is well-defined, note we must "avoid redundancy" in the decimal expansions by e.g. always using the decimal expansion of x that isn't eventually 9 as we did in class. It's not too hard to see, but useful to try and show, that this f is injective; but it's *not* surjective (why not?).