Math 215 - Solutions to Midterm 1 Review

October 4, 2018

1. Give a truth table for each of the following propositional forms:

(a) $P \to (P \vee Q)$

(b) $P \wedge (P \leftrightarrow \neg Q)$

(c) $(P \vee Q) \to \neg R$

Are any of these tautologies? Contradictions?

(a)

P	Q	$P \vee Q$	$P \to (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(b)

P	Q	$P \leftrightarrow \neg Q$	$P \wedge (P \leftrightarrow \neg Q)$
T	T	F	F
T	F	T	T
F	T	T	F
F	F	F	F

(c)

P	Q	R	$(P \vee Q)$	$\neg R$	$(P \vee Q) \to \neg R$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

(a) is a tautology; (b) and (c) are neither tautologies nor contradictions.

2. Prove by contradiction that if xy is even, then x is even or y is even.

Suppose not. Then xy is even, but both x and y are odd. Let a, b be integers such that $x = 2a + 1$ and $y = 2b + 1$. Then

$$xy = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1.$$

So $xy = 2k + 1$ with $x = 2ab + a + b$; then xy is odd. This contradicts our assumption that xy is even.

3. Show that if $x \in \mathbb{R}$ with $x \geq -1$, then for all $n \in \mathbb{N}$,

$$1 + nx \leq (1 + x)^n.$$

Where did you use the assumption that $x \geq -1$?

We prove this by induction on n.

Our base case is $n = 0$. Then $1 + nx = 1 + 0x = 1$ and $(1 + x)^n = (1 + x)^0 = 1$. So we have the base case.

Suppose inductively that we have some $k \in \mathbb{N}$ with $(1+kx) \le (1+x)^k$. We now use the fact that $x \ge -1$: This gives $1+x \ge 0$. Thus we can multiply through without changing the direction of the inequality:

$$(1+kx)(1+x) \le (1+x)^{k+1}$$

The left hand side is equal to $1+(k+1)x+kx^2$. Since $kx^2 \ge 0$, we have

$$1+(k+1)x \le 1+(k+1)x+kx^2 = (1+kx)(1+x) \le (1+x)^{k+1}.$$

Thus (by transitivity of $\le$) we have the inequality in the case $n = k+1$.

By the principle of induction, we have $1+nx \le (1+x)^n$ for all $x \ge -1$ and $n \in \mathbb{N}$.

4. Show that for all integers $n \ge 1$:

$$\sum_{i=1}^{n} 2i-1 = n^2.$$

Proof. By induction. Base case: $n = 1$.

$$\sum_{i=1}^{1} 2i-1 = 2-1 = 1 = 1^2.$$

For the inductive step, assume for some $k \ge 1$:

$$\sum_{i=1}^{k} 2i-1 = k^2.$$

Then:

$$\begin{aligned} \textstyle\sum_{i=1}^{k+1} 2i-1 \quad &= \left(\textstyle\sum_{i=1}^{k} 2i-1\right) + 2(k+1)-1 \text{ (definition of sum)} \\ &= k^2+2(k+1)-1 \text{ (inductive hypothesis)} \\ &= (k+1)^2 \end{aligned}$$

as needed. By the principle of induction, we have the identity for all $n \ge 1$. $\square$

5. Show that for all sets A, B, if $A \subseteq B$ then $B^c \subseteq A^c$.

Assume A, B are sets with $A \subseteq B$. That is, for all x, $x \in A$ implies $x \in B$. We need to show $B^c \subseteq A^c$. So suppose $x \in B^c$, that is, $x \notin B$. By contrapositive, we have $x \notin A$, that is, $x \in A^c$. This shows $B^c \subseteq A^c$ as claimed.

6. Let A_1, A_2, A_3 be sets. Show there are sets B_1, B_2, B_3 such that for i with $1 \le i \le 3$, we have $B_i \subseteq A_i$, $A_1 \cup A_2 \cup A_3 = B_1 \cup B_2 \cup B_3$ and the B_i are *disjoint*: that is, $B_i \cap B_j = \varnothing$ whenever $i \ne j$.

We define $B_1 = A_1$, $B_2 = A_2 - A_1$, and $B_3 = A_3 - (A_1 \cup A_2)$. Clearly if $x \in B_i$ we must have $x \in A_i$, by definition of set difference; so $B_i \subseteq A_i$ for $i \in \{1, 2, 3\}$.

We need to show we obtain the same union. $B_1 \cup B_2 \cup B_3 \subseteq A_1 \cup A_2 \cup A_3$, since by the previous paragraph, each A_i is a subset of B_i. We need to show the reverse inclusion.

Suppose $x \in A_1 \cup A_2 \cup A_3$. We have several cases.

If $x \in A_1$, clearly $x \in B_1$, hence $x \in B_1 \cup B_2 \cup B_3$.

So suppose $x \notin A_1$; then $x \in A_2 \cup A_3$. If $x \in A_2$, then $x \in A_2 - A_1 = B_2 \subseteq B_1 \cup B_2 \cup B_3$ as needed.

Finally, suppose $x \notin A_1$ and $x \notin A_2$; that is, $x \notin A_1 \cup A_2$. We must have $x \in A_3$. Hence $x \in A_3 - (A_1 \cup A_2) = B_3 \subseteq B_1 \cup B_2 \cup B_3$.

This completes the proof that $A_1 \cup A_2 \cup A_3 = B_1 \cup B_2 \cup B_3$.

All that is left to show is that the sets are disjoint. If $x \in B_3$, then $x \notin A_1 \cup A_2$, and so $x \notin B_1 \cup B_2$ (since $B_i \subseteq A_i$ means $x \notin A_i$ implies $x \notin B_i$). So the sets $B_1 \cap B_3$ or $B_2 \cap B_3$ are empty. Similarly, if $x \in B_2$, then $x \notin A_1 = B_1$; so $B_1 \cap B_2 = \varnothing$ as needed. Since these are all possible pairs of the sets B_1, B_2, B_3, we have shown these are pairwise disjoint.

7. Give a translation of each of the following in plain English; determine whether each is true or false.

(a) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{N}\, x < y$

(b) $\exists y \in \mathbb{Z}\, \forall z \in \mathbb{R}\, y < z \cdot z$

(c) $\exists x \in \mathbb{R}\, \exists y \in \mathbb{R}\, y \neq 0 \wedge x + y = x$

(d) $\forall x \in \mathbb{N}\, \forall y \in \mathbb{N}\, \exists z \in \mathbb{N}\, x < y \to (x < z \wedge z < y)$

(a): "For all reals x, there is a natural number y greater than x." This is true; if $x < 0$ then $y = 0$ works, and if $x \geq 0$ then there is some natural number $y \in (x, x+1]$, so $x < y$.

(b): "There is an integer y that is less than z^2, for all reals z." This is true, since $-1 < 0 \leq z^2$ for all reals z.

(c): "There are reals x, y such that y is nonzero and $x + y = x$." This is false; if we had $x + y = x$ for any reals x, y, then subtracting x on both sides gives us $y = 0$.

(d): "For all pairs of naturals x, y, there exists some natural z so that if $x < y$, we have $x < z < y$." Or: "For naturals $x < y$, there exists a third natural z strictly between them." It's false: $x = 0, y = 1$ gives a counterexample.

8. Using mathematical symbols ($+$, $\cdot$, quantifiers, etc.) only, give a definition of the predicate $P(n)$: "n is prime."

n is prime iff it has exactly two divisors, 1 and n. To avoid the case $n = 1$, we can say: n is prime iff $n > 1$ and whenever $k \in \mathbb{N}$ divides n, either $k = 1$ or $k = n$. We just need to expand the definition of "k divides n": $\exists m \in \mathbb{N}\, m \cdot k = n$. Putting all this together:

$$n > 1 \wedge \forall k \in \mathbb{N}\, \forall m \in \mathbb{N}\, m \cdot k = n \to (k = 1 \vee k = n).$$

Why are both of these quantifiers universal? Note that if we just plugged in the definition of "k divides n" in our definition of primeness above, we'd get

$$n > 1 \wedge \forall k \in \mathbb{N} (\exists m \in \mathbb{N}\, m \cdot k = n) \to (k = 1 \vee k = n).$$

(Note the parentheses.) Since $P \to Q$ is the same as $\neg P \vee Q$, this is equivalent to

$$n > 1 \wedge \forall k \in \mathbb{N} \neg(\exists m \in \mathbb{N}\, m \cdot k = n) \vee (k = 1 \vee k = n).$$

Applying the rules for negating quantifiers, and the tautology $(P \to Q) \leftrightarrow (\neg P \vee Q)$ again, gives our original answer above.

Note that replacing "$\forall m$" by "$\exists m$" in that original answer gives a predicate that is true for *all* $n > 1$ (and so does not define primeness); can you see why?

9. The *twin prime conjecture* states: "there are infinitely many primes p such that $p + 2$ is also prime."

Using symbols, give a statement of the twin prime conjecture. (You may use $P(n)$ as an abbreviation for the predicate you gave in the previous problem.)

$$\forall x \in \mathbb{N}\, \exists p \in \mathbb{N}\, x < p \wedge P(p) \wedge P(p+2).$$

This is because "there are infinitely many" for naturals is the same as "there are arbitrarily large" ("for all x, there is some larger y..."