Math 215 - Worksheet 3

September 26, 2018

1. Consider the following putative definitions of a function $f : \mathbb{R} \to \mathbb{R}$. Which of these is well-defined? For those that aren't, what about the definition would you change to make it "well"?

(a) $f(x) = \text{some } y \in \mathbb{R} \text{ with } y^2 = x$

(b) $f(x) = \text{some } y \in \mathbb{R} \text{ with } y > 0 \text{ and } e^y = x$

(c) $f(x) = \text{some } y \in \mathbb{R} \text{ with } \tan^{-1}(y) = x$

(d) $f(x) = \text{some } y \in [-\pi, \pi] \text{ with } \sin(y) = x$

Recall if $\mathrm{dom}(f) = A$, we define $\mathrm{Im}(f) = \{y \mid (\exists x \in A) f(x) = y$.

2. Let $f : A \to B$ and $g : B \to C$. Show $\mathrm{Im}(g \circ f) \subseteq \mathrm{Im}(g)$.

3. Show by example that proper inclusion "$\subsetneq$" is possible in the previous problem.

A function $f : A \to B$ is *one-to-one* (or *injective*) if for all $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.
A function $f : A \to B$ is *onto* (or *surjective*) if for all $b \in B$, there is some $a \in A$ with $f(a) = b$.

4. State the definitions of one-to-one and onto function symbolically (that is, using quantifiers and boolean connectives rather than words).

5. What do you get taking the contrapositive in the definition of "one-to-one"?

6. Show $f : A \to B$ is onto iff $\mathrm{Im}(f) = B$.

7. Show $f : A \to B$ is injective if and only if for all subsets $X, Y \subseteq A$, we have

$$\mathrm{Im}(f|(X \cap Y)) = \mathrm{Im}(f|X) \cap \mathrm{Im}(f|Y).$$