Name (Print): Solutions

NetID:

Math 210: Introduction to Advanced Mathematics

Midterm #1

October 5, 2018: 10-10:50am

- This exam contains 7 pages and 9 problems worth a total of 100 points. Please check to make sure your exam contains all pages and problems!
- No notes, books, calculators or other electronic devices should be out at any point during the exam. Phones should be off and put away.
- Write all proofs in complete sentences. Be sure to define any notation you introduce if it wasn't used in class. If proving a statement by induction, carefully state what you are proving, and what your inductive hypothesis is. If necessary, draw a box around your proof to separate it from any scratch work.
- Good luck!

1. (10 points) Using the provided grid, give a truth table for the propositional form:
$P \to (Q \to (P \wedge Q))$

+2 setup

P	Q	$P \wedge Q$	$Q \to P \wedge Q$	$P \to (Q \to (P \wedge Q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

+1 each fully correct row

Is this a tautology, contradiction, or neither?

+4 if matches last column

tautology

2. (5 points) Fill in the set-builder notation below so that A is the set of *rational* roots of the polynomial $p(x) = 4x^3 - 3x^2 - 8x + 6$:

$$A = \{x \in \mathbb{Q} \mid 4x^3 - 3x^2 - 8x + 6 = 0$$

3. (12 points) Give a complete definition of each of the following notions.

(a) $B \subseteq C$ (give a definition, not just the word)

B is a subset of C:
For all $x \in B$, we have $x \in C$.

(b) additive inverse

The additive inverse of a is $-a$.
or:
For all x there is a unique y s.t. $x + y = 0$; y is additive inverse of x

(c) graph of $f : A \to B$

$\{\langle x, f(x) \rangle \mid x \in A\}$

4. (12 points) Below are several claims with attempts at proofs. For each "proof", decide whether the argument is correct; if it is incorrect, say precisely what the error in reasoning is.

(a) **Claim.** For all sets a, $\varnothing \subseteq a$.

Proof. For all sets a, either $\varnothing \subseteq a$ or $\varnothing \not\subseteq a$. Suppose for a contradiction that $\varnothing \not\subseteq a$. Then there must exist some $x \in \varnothing - a$. In particular $x \in \varnothing$. But this contradicts the definition of $\varnothing$ as the set with no elements. □

3

Correct.

(b) **Claim** For all reals x, $|x - 7| < x^2 - 7$.

Proof. We have for $x = 5$, that $|5 - 7| = 2 < 18 = 5^2 - 7$. This proves the claim for reals x. □

3

Incorrect. Proves existence of such an x, not that claim holds for all x.

(c) **Claim.** Let $x \in \mathbb{N}$. If $x + 13$ is composite, then x is prime.

Proof. We show the contrapositive. So suppose x is prime. We have the cases $x = 2$ or $x \neq 2$. If $x = 2$ then $x + 13 = 15$, which is composite. If $x \neq 2$ then because x is prime, x is odd. So $x + 13$ is a sum of two odds, and so is even. This proves $x + 13$ is composite if x is prime. □

3

Incorrect. The proof is of the converse, not the contrapositive.

(d) **Claim.** For all functions $f : \mathbb{R} \to \mathbb{R}$, if f is differentiable, then f is continuous.

Proof. Suppose for a contradiction that for all functions $f : \mathbb{R} \to \mathbb{R}$, if f is differentiable then f is not continuous. But then we have that $f(x) = x^2$ is a function which is both differentiable and continuous, a contradiction. This contradiction gives us that all differentiable functions are continuous, as desired. □

3

Incorrect.

Desired claim: $\forall f \, (f \text{ dif} \to f \text{ cts})$

Negation would be: $\exists f \, (f \text{ dif} \wedge \neg(f \text{ cts}))$

But the "proof" gives $\forall f \, \neg(f \text{ dif} \to f \text{ cts})$ as the negation.

5. (20 points) Show for all naturals $n \geq 7$ that $n^2 + 4 > 7n$.

Proof: We show this by induction.

Base case: $n = 7$.

Then $n^2 + 4 = 49 + 4 = 53 > 49 = 7n$.

This shows the base case.

For the inductive step, we may assume: $k^2 + 4 > 7k$ for some $k \geq 7$, $k \in \mathbb{N}$.

We have:

$$(k^2 + 4) > 7k$$
$$\Rightarrow k^2 + 2k + 1 + 4 > 7k + 2k + 1$$
$$\Rightarrow (k+1)^2 + 4 > 9k + 1$$

$9k + 1 = 7k + 1 + 2k$; since $k \geq 7$, $1 + 2k \geq 7$.

So $9k + 1 > 7k + 7 = 7(k+1)$.

Putting these together:

$$(k+1)^2 + 4 > 7(k+1).$$

This completes the inductive step.

By induction, we have shown the claim for all naturals $n \geq 7$.

6. (14 points) (a) Show $\{12x - 3 \mid x \in \mathbb{Z}\} \subseteq \{n \in \mathbb{Z} \mid 3 \text{ divides } n\}$.

We need to show: if $n = 12x - 3$ with $x \in \mathbb{Z}$, then n is a multiple of 3.

Suppose $n = \cancel{n}\,12x - 3$. Then $n = 3(4x - 1)$. Since $4x - 1 \in \mathbb{Z}$, we have that 3 divides n as needed.

(b) Show that part (a) is false if the inclusion "$\subseteq$" is replaced with "=".

For the equality to hold we would just need "$\supseteq$". So we just need that $\not\supseteq$ in the above. That is: there is n divisible by 3, that is not of the form $12x - 3$ for any $x \in \mathbb{Z}$.

So take $n = 6$. Then if

$$12x - 3 = n,$$

have

$$12x = 9 \Rightarrow x = \frac{4}{3} \notin \mathbb{Z}.$$

So 6 is in 2nd set, not in first.

7. (12 points) In each part of this problem, you are given an attempted definition of a function. For each attempt, determine whether or not it yields a well-defined function. If it does not, explain why not.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) =$ the least $y \in \mathbb{N}$ such that $x < y$.

2 Well-defined.

(b) $g : \mathbb{R} \to \mathbb{R}$, $g(x) =$ some $n \in \mathbb{Z}$ with $n < x < n + 1$.

4 Not well-defined. There is no such n when x is an integer.

(2 for explanation)

(c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $h(x, y) = x^y$.

Not-well-defined.

4 If $(x,y) = (2,-1)$, then $x^y = 2^{-1} = \frac{1}{2}$, which is not in the codomain $\mathbb{Z}$.

(2 for explanation)

(d) $i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$, $i(x, y) =$ some $n \in \mathbb{N}$ with $x - y = n$ or $y - x = n$.

This is well-defined.

2 If $x=y$ then $x-y=y-x=0 \in \mathbb{N}$;

If $x<y$ then $y-x \in \mathbb{N}$, and $x-y \notin \mathbb{N}$.

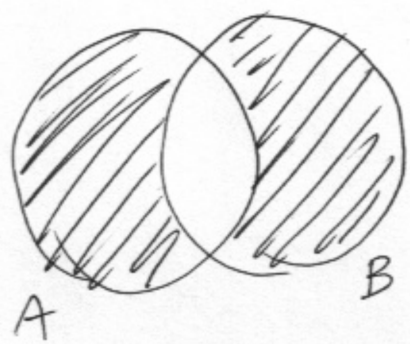
If $y<x$, $x-y \in \mathbb{N}$ $y-x \notin \mathbb{N}$.

So output is unique for all pairs of integers x,y.

For the next two problems, define the *symmetric difference* of sets A and B, written $A \mathbin{\Delta} B$,

$$A \mathbin{\Delta} B := (A - B) \cup (B - A).$$

8. (3 points) Draw a Venn diagram of A and B, with $A \mathbin{\Delta} B$ shaded in.



9. (12 points) Decide whether each stated relationship is true for all sets A, B or not:

(a) $\forall x \in A\ (x \in B \to x \notin A \mathbin{\Delta} B)$

2 each.

True.

(b) $A \in A \mathbin{\Delta} B$

False.

(c) $(A \mathbin{\Delta} B) \cap (B - A) = \varnothing$

False.

(d) $(A - B) \subseteq A \mathbin{\Delta} B$

True.

(e) $A \mathbin{\Delta} B \subseteq A^c \cap B^c$

False

(f) $A \mathbin{\Delta} B = (A \cup B) - (A \cap B)$

True