Name (Print): Solutions

NetID:

Math 210: Introduction to Advanced Mathematics

Midterm #1

October 5, 2018: 4-4:50pm

- This exam contains 7 pages and 9 problems worth a total of 100 points. Please check to make sure your exam contains all pages and problems!
- No notes, books, calculators or other electronic devices should be out at any point during the exam. Phones should be off and put away.
- Write all proofs in complete sentences. Be sure to define any notation you introduce if it wasn't used in class. If proving a statement by induction, carefully state what you are proving, and what your inductive hypothesis is. If necessary, draw a box around your proof to separate it from any scratch work.
- Good luck!

1. (10 points) Using the provided grid, give a truth table for the propositional form: $Q \wedge (P \leftrightarrow (\neg P \wedge Q))$

+1 each fully correct row

P	Q	$\neg P$	$\neg P \wedge Q$	$P \leftrightarrow (\neg P \wedge Q)$	$Q \wedge (P \leftrightarrow (\neg P \wedge Q))$
T	T	F	F	F	F
T	F	F	F	F	F
F	T	T	T	F	F
F	F	T	F	T	F

Is this a tautology, contradiction, or neither?

+4 if matches final column, above

contradiction

2. (5 points) Fill in the set-builder notation below so that A is the set of *positive real* roots of the polynomial $p(x) = 4x^3 - 3x^2 - 8x + 6$:

$$A = \{x \in \mathbb{R} \mid x > 0 \text{ and } p(x) = 0\}$$

or $\mathbb{R}^{>0} \mid p(x) = 0$, etc.

3. (12 points) Give a complete definition of each of the following notions.

(a) $B \times C$ (give a definition, not just the word)

Cartesian product of B and C:

$$B \times C = \{\langle b, c \rangle \mid b \in B \text{ and } c \in C\}$$

(b) additive identity

0 (zero)

or: the unique x such that for all reals a, $a + x = a = x + a$

(c) range of $f : A \to B$

$$\{y \in B \mid y = f(x) \text{ for some } x \in A\}$$

or

$$\{f(x) \mid x \in A\}$$

4. (12 points) Below are several claims with attempts at proofs. For each "proof", decide whether the argument is correct; if it is incorrect, say precisely what the error in reasoning is.

   (a) **Claim.** For all sets a, $\varnothing \subseteq a$.

   false

   Proof. Fix a set a. Clearly we have either $\varnothing \subseteq a$ or $a \subseteq \varnothing$. By the definition of $\varnothing$ as the empty set, we have $a \notin \varnothing$, therefore $a \not\subseteq \varnothing$. By cases, we conclude that $\varnothing \subseteq a$. □

   doesn't follow.

3

Incorrect.

   (b) **Claim.** For all integers x, if $x = 10k - 5$, then 5 divides x.

   Proof. Suppose x is an integer such that $x = 10k - 5$ for some $k \in \mathbb{Z}$. Then $x = 5(2k-1)$, and it follows that 5 divides x, completing the proof. □

3

Correct

   (c) **Claim** For all reals x, $|x-7| < x^2 - 7$.

   Proof. We have the cases $x \geq 7$ and $x < 7$. If $x \geq 7$ then $7x \leq x^2$, and we have $|x-7| = x - 7 < 7x - 7 \leq x^2 - 7$. In the other case that $x < 7$, we have $x - 7 < 0$ and since $x^2 > 0$, we get $|x-7| < x^2 - 7$. In either case we obtain the desired inequality. □

3

Incorrect. The underlined inference does not follow from previous inequalities.

   (d) **Claim.** For all functions $f : \mathbb{R} \to \mathbb{R}$, if f is differentiable, then f is continuous.

   Proof. Suppose for a contradiction that for all functions $f : \mathbb{R} \to \mathbb{R}$, if f is differentiable then f is not continuous. But then we have that $f(x) = x^2$ is a function which is both differentiable and continuous, a contradiction. This contradiction gives us that all differentiable functions are continuous, as desired. □

3

Incorrect.

Desired claim: $\forall f \quad f \text{ dif} \to f \text{ cts}$

Negated, this is:

$\exists f \quad f \text{ dif} \wedge \neg(f \text{ cts})$

So the assumption for a contradiction is not actually the negation of the claim.

5. (20 points) Show for all naturals $n \geq 4$ that $3^n > 2n^2 + 3n$.

We prove this by induction.

In the base case, $n=4$. We have

$$3^n = 3^4 = 81$$

$$2n^2 + 3n = 44$$

Since $81 > 44$, we have base case.

For the inductive step, we assume that

$$3^k > 2k^2 + 3k$$

for some $k \in \mathbb{N}$ with $k \geq 4$. Our goal is to show

$$3^{k+1} > 2(k+1)^2 + 3(k+1)$$
$$= 2k^2 + 4k + 2 + 3k + 3$$
$$= 2k^2 + 7k + 5$$

So we multiply our inductive hypothesis by 3, and obtain

$$3 \cdot 3^k > 6k^2 + 9k$$
$$\rightarrow 3^{k+1} > (2k^2 + 7k + 5) + 4k^2 + 2k - 5$$
$$\rightarrow 3^{k+1} > 2(k+1)^2 + 3(k+1) + (4k^2 + 2k - 5)$$

Since $k \geq 4$, we have

$$4k^2 + 2k - 5 \geq 64 + 8 - 5 > 0,$$

so we have shown

$$3^{k+1} > 2(k+1)^2 + 3(k+1).$$

By induction, we have shown the claim for all $n \in$ $n \geq 4$.

6. (14 points) (a) Show $\{x + 13 \mid x \text{ is prime}\} \subseteq \{n \in \mathbb{N} \mid n \text{ is composite}\}$.

We just need to show: if $n = x+13$ with x prime, then n is composite.

Case 1: $x=2$. Then $n = x+13 = 2+13 = 15$, which is compos

Case 2: $x \neq 2$. Then x prime means x is odd, $x>2$.
Then $n = x+13 = (\text{odd} + \text{odd})$, so n is even and greater than 2. So we have $n = 2k$ some $k>1$, and n is composite.

(b) Show that part (a) is false if the inclusion "⊆" is replaced with "=".

This is the same as showing that ~~#~~ "⊇" does not hold. That is, there is some composite n that is not equal to $x+13$ for any prime x.

For example: if $n=21$, then $n = 3 \cdot 7$ is com
But if $x+13 = 21$, then $x = 8$ which is not prime.

7. (12 points) In each part of this problem, you are given an attempted definition of a function. For each attempt, determine whether or not it yields a well-defined function. If it does not, explain why not.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) =$ the largest $y \in \mathbb{N}$ such that $y < x$.

4

Not well-defined.

If $x \leq 0$, then there is <u>not</u> any $y \in \mathbb{N}$ with $y < x$.

(2 for explanation)

(b) $g : \mathbb{R} \to \mathbb{Z}$, $g(x) = n$ such that $n \leq x < n + 1$.

2

Well-defined.

(every real x is in $[n, n+1)$ for some unique integer, n.)

(c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $h(x, y) = |x - y|$.

2

Well-defined.

(d) $i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$, $i(x, y) =$ the least n with $x < n$ and $n < y$.

Not well-defined.

4

There needn't be any such n.

E.g. if $(x,y) =$ $(1, -1)$

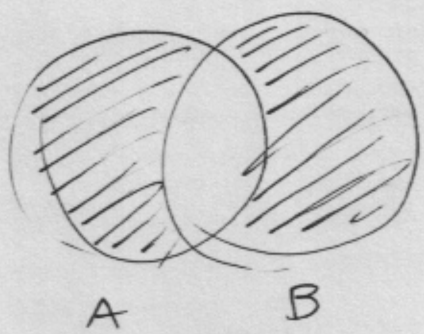
(2 for explanation)

there is no n with $1 < n$, $n < -1$.

For the next two problems, define the *symmetric difference* of sets A and B, written $A \Delta B$,

$$A \Delta B := (A \cup B) - (A \cap B)$$

8. (3 points) Draw a Venn diagram of A and B, with $A \Delta B$ shaded in.



9. (12 points) Decide whether each stated relationship is true for all sets A, B or not:

(a) $\exists x \in A\ (x \notin A \Delta B)$

False. (could have $A \cap B = \emptyset$.)

(b) $A \Delta B = B \Delta A$

True.

(c) $(A \Delta B) \cap (A \cap B) = \emptyset$

True.

(d) $(A - B) \subseteq A \Delta B$

True.

(e) $A \Delta B \subseteq A^c \cap B^c$

False.

(f) $A \Delta B = (A - B) \cup (B - A)$.

True.