Solutions to Assignment 4

Math 300 – Spring 2019 (Hachtman)

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Proposition 1. Let B be the relation on the set of people: $x\,B\,y$ iff x and y have the same birthday.

Then B is an equivalence relation.

Proof. It is clearly reflexive, as every person has one and only one birthday, which is clearly the same birthday as their own birthday!

If x and y are people that have the same birthday, then of course y and x have the same birthday; so B is symmetric.

Finally, if x and y both have the same birthday, and y and z both have the same birthday, then all three share the same birthday (since everyone has exactly one birthday); so $x\,B\,z$. This shows B is transitive.

We have verified the three defining properties of being an equivalence relation. □

Proposition 2. Let C be a relation on the set of people, defined by $x\,C\,y$ iff x and y have citizenship in the same country.

Proof. C is *not* an equivalence relation. The point is that, unlike the case with birthdays, someone need not have citizenship in a *unique* country (there is such a thing as *dual* citizenship). We will see how it follows that C is not transitive.

I am a U.S. citizen. Actress Salma Hayek, known for her roles in such films as *Desperado*, *Frida* and *Wild Wild West*, is a dual Mexican-American citizen; so Salma Hayek and I have citizenship in the same country: Sherwood C Salma Hayek.

Also, director Guillermo del Toro (*Hellboy*, *Pacific Rim*) is a sole citizen of Mexico. So Salma Hayek C Guillermo del Toro.

But Sherwood $\not C$ Guillermo del Toro. So C is not transitive.

(Note that C is not even reflexive, as can be seen by the existence of *stateless persons* who have no recognized nationality; for example, Albert Einstein was a stateless person from 1896 to 1901.) □

Proposition 3. Let, for $x, y \in \mathbb{R}$, $x\,R\,y$ iff $x - y$ is a rational number.

Then R is an equivalence relation.

Proof. R is reflexive: for any real number, $x - x = 0 \in \mathbb{Q}$.

R is symmetric: Suppose $x - y = q \in \mathbb{Q}$. Then $y - x = -(x - y) = -q$, which is rational.

Finally, we show R is transitive. Suppose x, y, z are reals with $x \, R \, y$ and $y \, R \, z$. Let $q = x - y$ and $r = y - z$; so q, r are rationals. Then

$$x - z = (x - y) + (y - z) = q + r.$$

Since $\mathbb{Q}$ is closed under addition, we have $q + r \in \mathbb{Q}$, so that $x \, R \, z$. This shows R is transitive.

Since R is reflexive, symmetric, and transitive, we have that R is an equivalence relation. □

Proposition 4. Let, for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \, S \, \mathbf{v}$ iff $\mathbf{u} \cdot \mathbf{v} = 0$. Then S is not an equivalence relation.

Proof. Much like the citizenship example, S is neither reflexive nor transitive (but is symmetric).

To see S is not reflexive, consider $\mathbf{u} = \langle 1, 2, 3 \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = \langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle = 1^2 + 2^2 + 3^2 = 15 \neq 0,$$

so $\mathbf{u} \not S \mathbf{u}$. (Note the same argument shows $\mathbf{u} \not S \mathbf{u}$ whenever $\mathbf{u}$ does not equal the zero vector $\mathbf{0} = \langle 0, 0, 0 \rangle$. □