Assignment 5: Primes and density

Math 300 – Spring 2019 (Hachtman)

Due Monday, April 8

1. A natural number n is a *divisor* of m if $kn = m$ for some natural number k. A number $p \in \mathbb{N}$ is *prime* if it has exactly two divisors: 1 and itself.

   $\boxed{\text{Prove that the set of primes is infinite.}}$

   You may want to use the outline here:

   - Show that if $n \neq 1$ and n divides m, then n does not divide $m + 1$.
   - Show every natural number $m > 1$ has at least one prime divisor.
   - Suppose that the set of primes is finite. Let m be the product of all of them, and consider $m + 1$ to obtain a contradiction.

2. Recall we defined the density of $A \subseteq \mathbb{N}$ to be the value of the limit

   $$\rho(A) := \lim_{n\to\infty} \frac{|A \cap [0, n]|}{n + 1}.$$

   However, the limit does not always exist!

   $\boxed{\text{Define a set } A \text{ where the limit in the definition of density } \rho(A) \text{ does not exist, and prove it.}}$

   Hints:

   - Your goal is to define A. For any increasing sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ of natural numbers, you can define A one piece at a time, by deciding what $A \cap [n_k, n_k + 1)$ will be.
   - Suppose that

     $$\frac{|A \cap [0, n]|}{n + 1}$$

     is very very small. Show there is a number m so that

     $$\frac{|(A \cap [0, n]) \cup [n + 1, m]|}{m + 1}$$

     is very very close to 1.

Write up your solutions in LaTeX and submit them by email as usual.