Applied Linear Algebra
Instructor: Hachtman
Midterm 2 Grading Rubric
4/7/2017

You have 50 minutes to complete this exam. No notes, calculators, phones etc. are permitted. **You must SHOW ALL YOUR WORK to receive credit!**

1. (20 points) Determine whether each of the following sets is a vector subspace of $\mathbf{R}^n$ for n as shown. If it is a subspace, show that it is a subspace. If it is not, explain why.

   (a) (10 points) $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbf{R}^3 \;\middle|\; a+b+c=0 \right\}$. The point is to verify the three properties

   (1) V contains $\mathbf{0}$,
   (2) V is closed under vector addition,
   (3) V is closed under scalar multiplication.

   3 points awarded (1 per property) for *mentioning* each property.
   If all properties are mentioned, award 1 additional point.
   6 points awarded (2 per property) for *correctly* verifying. For example, "suppose c is a scalar and $\mathbf{x} \in V$. Then $x_1+x_2+x_3=0$, and $c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$. Since $cx_1+cx_2+cx_3 = c(x_1+x_2+x_3) = 0$, we have $c\mathbf{x} \in V$. This shows (3)." (Less detail is OK.)

   (b) (10 points) $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is non-invertible} \right\}$.

   Here the answer is no, because W is not a subspace. Full credit for saying why and giving a counterexample, e.g.: "$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are in W since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both non-invertible. But $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is not in W since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is* invertible. So W is not closed under vector addition, and so is not a subspace"
   Partial credit awarded as follows:
   +1 point for correctly showing W contains the zero vector.
   +1 point for correctly showing W is closed under scalar multiplication.
   +3 points for saying that W is not closed under vector addition, but with a faulty explanation.

2. (15 points) Let $\mathcal{B} = \{5t^2 + 4, t + 2, t^2 - 2t\}$ and $\mathcal{C} = \{1, t + 1, t^2 + t + 1\}$. Then $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbf{P}_2$. Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.

+3 points for setting up the problem by putting matrices C, B corresponding in some way to $\mathcal{C}$ and $\mathcal{B}$ side-by-side. (For example as a partitioned matrix, $[C|B]$). (It's fine if the order is switched, as long as the row reduction is done to make the C-side into I; if not, award 1 point on this part.)

+2 if the matrices B, C above are 3×3.

+4 if the set-up $[C|B]$ is otherwise correct. Two examples of correct setups would be

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 5 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -2 \\ 1 & 1 & 1 & 4 & 2 & 0 \end{array}\right], \text{ or } \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 5 & 0 & 1 \end{array}\right]$$

(-1 for each small error, e.g. confusing entries)

+4 for row-reducing correctly, -1 for each arithmetic mistake.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 1 & 2 \\ 0 & 1 & 0 & -5 & 1 & -3 \\ 0 & 0 & 1 & 5 & 0 & 1 \end{array}\right]$$

+2 for giving the final answer,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 4 & 1 & 2 \\ -5 & 1 & -3 \\ 5 & 0 & 1 \end{bmatrix}$$

3. (15 points) Consider the matrix A with row echelon form rref(A):

$$A = \begin{bmatrix} 0 & -8 & 8 & 40 & 8 \\ 0 & -6 & 0 & 18 & 6 \\ 0 & 0 & 10 & 20 & 10 \\ 0 & 1 & 3 & 3 & 0 \end{bmatrix}, \qquad \text{rref}(A) = \begin{bmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Give a basis for each of the following subspaces associated with A.

(a) (4 points) Col(A) This should just be the set of pivot columns from A,

$$\left\{ \begin{bmatrix} -8 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 10 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \\ 10 \\ 0 \end{bmatrix} \right\}.$$

If listed the pivots of rref(A) instead, only get 1 point.
No partial credit for otherwise different answers.

(b) (7 points) Nul(A)
+1 for knowing you're solving $A\mathbf{x} = \mathbf{0}$ (this could be indicated by just writing down equations involving x_1, x_2, etc.)
+1 for saying "x_1, x_4 are free" or indicating some other way you know these are free.
+2 for correctly solving $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} x_2 - 3x_4 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_5 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_2 &= 3x_4 \\ x_3 &= -2x_4 \\ x_5 &= 0 \end{aligned}$$

+2 for finding the solution in parametric form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_4 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

+1 for writing the basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(c) (4 points) Row(A) This is just the rows of rref(A), which can be written as row or column vectors, that is

$$\{[0, 1, 0, -3, 0], [0, 0, 1, 2, 0], [0, 0, 0, 0, 1]\}$$

or

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

But they should be vectors: -1 point if they're written the wrong way, e.g. $\{0, 1, 0, -3, 0\}$ is not a vector. (And 0 credit for listing rows from A!)

4. (20 points) The eigenvalues of the matrix

$$B = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

are 1 and 2.

(a) (12 points) Find the dimensions of the eigenspaces E_1 and E_2.

The dimension of the eigenspace E_λ is $\dim(\text{Nul}(B - \lambda I)) = 3 - \text{rank}(B - \lambda I)$.

Here

$$B - I = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 2, so $\dim(E_1) = 1$. And

$$B - 2I = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 2, so $\dim(E_2) = 1$ as well.

For each eigenspace,

+1 point for looking at $B - \lambda I$.

+1 point for correctly row reducing.

+1 point for knowing to look at $3 - \text{rank}(B - \lambda I)$ or counting the number of free variables.

+3 points for giving the final answer as a number between 0 and 3. (Partial credit: +1 point for giving a basis instead of the dimension.)

(b) (8 points) Is B diagonalizable? Explain your answer.

No. An $n \times n$ matrix B is diagonalizable if and only the sum of the dimensions of its eigenspaces is equal to n. Here $n = 3$ and there are two eigenspaces, each with dimension 1. Since $1 + 1 = 2 \neq 3$, B is not diagonalizable.

Opportunities for partial credit:

+1 for *correctly and completely* defining diagonalizability

+2 for sensible statements involving the need for a basis of eigenvectors

+2 for sensible statements about needing the dimensions of the eigenspaces to add to 3

+1 for **each true, complete sentence**.

5. (15 points) The matrix

$$M = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

has complex eigenvalues.

(a) (6 points) Find the eigenvalues of M.

+2: Correctly setting up the characteristic polynomial: $(1-\lambda)(3-\lambda) - 5(-2)$

+1: Simplify and set equal to 0: $\lambda^2 - 4\lambda + 13 = 0$

+2: Knowing the quadratic formula, $\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

+1: Solve: $\frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$ (-1 each arithmetic mistake)

(b) (9 points) Find a basis for $\mathbf{C}^2$ consisting of (complex) eigenvectors of M.

+1: Set up the matrix $M - \lambda I$ using one of the λ values from part (a),

$$M - (2+3i)I = \begin{bmatrix} 1-2-3i & 5 \\ -2 & 3-2-3i \end{bmatrix} = \begin{bmatrix} -1-3i & 5 \\ -2 & -1-3i \end{bmatrix}$$

+4: Find the null space of this matrix by using a row to solve.

$$(-1-3i)x_1 + 5x_2 = 0; \text{ can set } x_1 = 5, x_2 = 1+3i.$$

This gives one vector, $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 1+3i \end{bmatrix}$.

+3: either repeat with the other eigenvalue, $2-3i$, or take the complex conjugate of $\mathbf{v}_1$, $\mathbf{v}_2 = \bar{\mathbf{v}}_1 = \begin{bmatrix} 5 \\ 1-3i \end{bmatrix}$.

+1: Put these together in a basis, e.g. $\left\{ \begin{bmatrix} 5 \\ 1+3i \end{bmatrix}, \begin{bmatrix} 5 \\ 1-3i \end{bmatrix} \right\}$.

6. (15 points) The matrix

$$Q = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$$

Has eigenvalues $4, 6$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find a solution to the differential equation

$$Q\mathbf{x}(t) = \mathbf{x}'(t)$$

satisfying $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

Remark: There's a mistake in this problem as written, as the order of eigenvectors is mixed up. But nobody will be penalized for this (or for noticing the mistake and switching them.)

+8: For having an answer of the form

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{6t}$$

some c_1, c_2. Penalties: -4 if the eigenvectors or eigenvalues don't appear this answer; -6 if the answer is syntactically nonsense (e.g. two vectors are multiplied together)

+7: Knowing that the coeffients c_1, c_2 are solved for by solving the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix},$$

and doing so correctly. (-2 for each arithmetic mistake, no partial credit if finding c_1, c_2 by incorrect method.)