Applied Linear Algebra
Instructor: Hachtman
Quiz 10 – 3/31/17

Name: ____________________

UIN: ____________________

This quiz has 2 pages, a front and a back! No notes, calculators, phones etc. are permitted. **Show all your work.**

1. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation such that $T(\mathbf{v})$ is the result of rotating $\mathbf{v}$ about the z-axis by an angle $\theta = \pi/3$; let A be the matrix of T.

   (a) (2 points) Find an eigenvector of A. What is the corresponding eigenvalue?

   Solution: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, 1.

   Explanation: An eigenvector of A is a vector $\mathbf{v} \in \mathbf{R}^3$ so that $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$. In particular, if $\mathbf{v}$ is an eigenvector of A, then $T(\mathbf{v}) = A\mathbf{v}$ is oriented in the same direction as $\mathbf{v}$. Here T is precisely a rotation of $\mathbf{R}^3$ about the z-axis by angle $\pi/3$. So the only vectors $\mathbf{v}$ that can be eigenvectors are those along the z-axis. So any vector $\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$ with $c \neq 0$ is an eigenvector; clearly $T(\mathbf{v}) = \mathbf{v}$ for all such $\mathbf{v}$, so the eigenvalue is 1.

   (b) (2 points) Is A diagonalizable? Explain your answer.

   Solution: No, A is not diagonalizable. By the diagonalization theorem, A is diagonalizable iff there is a basis for $\mathbf{R}^3$ consisting of eigenvectors for A. But this rotation changes the direction of any vector not along the z-axis, so A does not have any eigenvectors not along the z-axis, so there can be no such basis.

2. (3 points) Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$, where

$$A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}.$$

(The eigenvalues of A are $-1, 3$.)

Solution: Let $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$. Then P can be any 2×2 matrix whose columns are eigenvectors for $-1, 3$, respectively. So we find these:

$$A - (-1)I = \begin{bmatrix} -1 & -5 \\ 1 & 5 \end{bmatrix}$$

solving $-x_1 - 5x_2 = 0$, we have $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$ is a -1-eigenvector.

Next,

$$A - 3I = \begin{bmatrix} -5 & -5 \\ 1 & 1 \end{bmatrix}$$

and again solving the homogeneous system we get eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a 3-eigenvector.

So $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}$.

3. (3 points) The matrix $B = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ has complex eigenvalues. Find them.

Solution: The characteristic polynomial of B is $(3 - \lambda)(1 - \lambda) + 2 = \lambda^2 - 4\lambda + 5$. We use the quadratic formula to find the roots,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

So the eigenvalues are $2 + i$ and $2 - i$.