Applied Linear Algebra
Instructor: Hachtman
Quiz 11 – 4/14/17

Name: ____________________

UIN: ____________________

This quiz has 2 pages, a front and a back! No notes, calculators, phones etc. are permitted. **Show all your work.**

1. (4 points) Suppose A is a 2×2 real matrix and that $4 + 7i$ is an eigenvalue of A, with $\begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix}$ a corresponding eigenvector. Find a real solution to the differential equation

$$A\mathbf{x}(t) = \mathbf{x}'(t).$$

Solution:

We are given an eigenvalue λ and eigenvector $\mathbf{v}$, and we know that one solution is given by

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} = \begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix} e^{(2-3i)t}.$$

But this solution is not a real solution. We may simplify to separate this solution into real and complex parts, using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$:

$$\begin{aligned}
\begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix} e^{(2-3i)t} &= \begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix} e^{2t}e^{i(-3t)} \\
&= \begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix} e^{2t}(\cos(-3t) + i\sin(-3t)) \\
&= \begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix} e^{2t}(\cos(-3t) + i\sin(-3t)) \\
&= \left(\begin{bmatrix} 2(\cos(-3t) + i\sin(-3t)) - 3i(\cos(-3t) + i\sin(-3t)) \\ 1(\cos(-3t) + i\sin(-3t)) + 5i(\cos(-3t) + i\sin(-3t)) \end{bmatrix} \right) e^{2t} \\
&= \left(\begin{bmatrix} 2\cos(-3t) + 2i\sin(-3t) - 3i\cos(-3t) - 3i^2\sin(-3t) \\ \cos(-3t) + i\sin(-3t) + 5i\cos(-3t) + 5i^2\sin(-3t) \end{bmatrix} \right) e^{2t} \\
&= \left(\begin{bmatrix} 2\cos(-3t) + 2i\sin(-3t) - 3i\cos(-3t) + 3\sin(-3t) \\ \cos(-3t) + i\sin(-3t) + 5i\cos(-3t) - 5\sin(-3t) \end{bmatrix} \right) e^{2t} \\
&= \left(\begin{bmatrix} 2\cos(-3t) + 3\sin(-3t) \\ \cos(-3t) - 5\sin(-3t) \end{bmatrix} \right) e^{2t} + i\left(\begin{bmatrix} 2\sin(-3t) - 3\cos(-3t) \\ \sin(-3t) + 5\cos(-3t) \end{bmatrix} \right) e^{2t}
\end{aligned}$$

Now if $\mathbf{x}(t)$ is a complex solution, then $\mathrm{Re}(\mathbf{x}(t))$ and $\mathrm{Im}(\mathbf{x}(t))$ are *real* solutions. In fact, the general real solution is

$$c_1\left(\begin{bmatrix} 2\cos(-3t) + 3\sin(-3t) \\ \cos(-3t) - 5\sin(-3t) \end{bmatrix} \right) e^{2t} + c_2\left(\begin{bmatrix} 2\sin(-3t) - 3\cos(-3t) \\ \sin(-3t) + 5\cos(-3t) \end{bmatrix} \right) e^{2t}$$

and any such answer would be acceptable. (If you memorized the yellow box on page 318 of your book and didn't show any work, that's great. In fact, you could have given the smartass answer, letting $\mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all t, and this would be worth full credit.)

2. (6 points) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

(a) Find $\|\mathbf{v}\|$.
$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{1+9+4} = \sqrt{14}$.

(b) Find a vector $\mathbf{u} \in \text{span}\{\mathbf{v}\}$ with $\|\mathbf{u}\| = 1$.
$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} 1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$.

(c) Suppose $\mathbf{w}$ is a vector with $\|\mathbf{w}\| = 2$ and $\mathbf{v} \cdot \mathbf{w} = 0$. Find the distance between $\mathbf{w}$ and $\mathbf{v}$.
$\text{dist}(\mathbf{w}, \mathbf{v}) = \|\mathbf{w} - \mathbf{v}\| = \sqrt{(\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v})} = \sqrt{\mathbf{w} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v}} = \sqrt{\|\mathbf{w}\|^2 - 0 - 0 + \|\mathbf{v}\|^2} = \sqrt{4 + 14} = 3\sqrt{2}$