Applied Linear Algebra
Instructor: Hachtman
Quiz 12 – 4/21/17

Name: ______________________________

UIN: ______________________________

This quiz has 2 pages, a front and a back! No notes, calculators, phones etc. are permitted. **Show all your work.**

Recall: If $L = \text{span}\{\mathbf{u}\}$, then $\text{proj}_L(\mathbf{v}) = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{v}$.

1. (3 points) Fix a basis $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$.

   Solution: As usual, one could solve the equation $B\mathbf{y} = \mathbf{x}$ (where B is the matrix with columns the vectors of $\mathcal{B}$ and $\mathbf{y} = [\mathbf{x}]_{\mathcal{B}}$), but it's quicker to notice that $\mathcal{B}$ is an *orthogonal* basis, and therefore the entries y_i of the coordinate vector are

$$y_1 = \frac{\mathbf{x}\cdot\mathbf{b_1}}{\mathbf{b_1}\cdot\mathbf{b_1}} = \frac{0}{18} = 0$$
$$y_2 = \frac{\mathbf{x}\cdot\mathbf{b_2}}{\mathbf{b_2}\cdot\mathbf{b_2}} = \frac{3}{9} = 1/3$$
$$y_1 = \frac{\mathbf{x}\cdot\mathbf{b_3}}{\mathbf{b_3}\cdot\mathbf{b_3}} = \frac{6}{18} = 1/3$$

   Hence $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \end{bmatrix}$.

2. (4 points) Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Find an orthogonal basis for W.

Solution: This is just Gram-Schmidt (and is really the easiest kind, since we just need to find a second vector orthogonal to the first but still in the span). Hence

- $\mathbf{u_1} = \mathbf{w_1} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}$
- $\mathbf{u_2} = \mathbf{w_2} - \text{proj}_{\text{span}\{u_1\}}(\mathbf{w_2}) \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{19} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{19} \begin{bmatrix} 39 \\ -16 \\ 19 \\ 3 \end{bmatrix}$.

So one orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 39 \\ -16 \\ 19 \\ 3 \end{bmatrix} \right\}$ (note we made the answer slightly simpler by scaling $\mathbf{u_2}$ by 19, since this doesn't change orthogonality or being in W).

3. (3 points) Find the least-squares solution to the equation

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}.$$

(Recall least-squares solutions $\mathbf{x}$ satisfy $A^T A\mathbf{x} = A^T\mathbf{b}$.)

Solution: Computing $A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ we see that we must find solutions to

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

and a row reduction gives $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.