Applied Linear Algebra
Instructor: Hachtman
Quiz 3 – 1/27/17

Name: ____________________

UIN: ____________________

This quiz has 2 pages, a front and a back! No notes, calculators, phones etc. are permitted. **Show all your work.**

1. (4 points) For each of the following sets of vectors, say whether it is linearly independent or dependent, and give a brief explanation why.

(a) $\begin{bmatrix}2\\4\\6\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\\12\end{bmatrix}$

(b) $\begin{bmatrix}6\\5\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\12\\-9\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$

(c) $\begin{bmatrix}0\\5\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}1\\-5\end{bmatrix}$

(d) $\begin{bmatrix}22\\0\\0\end{bmatrix}, \begin{bmatrix}5\\13\\0\end{bmatrix}, \begin{bmatrix}0\\-6\\10\end{bmatrix}$

(a), (b), (c) are all linearly dependent. For (a), the second vector is 2 times the first. For (b) and (c), the number of vectors is larger than the dimension of the space, so these must be dependent.

(d) is linearly independent: Putting the vectors into a matrix we get one in row echelon form, and there's a pivot in every column.

2. (2 points) For each of the following, determine if the matrix-vector product makes sense, and if so, compute it.

(a) $\begin{bmatrix}1 & -3\\2 & 5\\0 & 3\end{bmatrix} \begin{bmatrix}1\\5\\0\end{bmatrix}$

This doesn't make sense, as the number of entries of the vector must match the number of columns in the matrix ($3 \neq 2$).

(b) $\begin{bmatrix}1 & 0 & 7\\-2 & 5 & 0\\0 & 3 & 11\end{bmatrix} \begin{bmatrix}3\\1\\-2\end{bmatrix} = \begin{bmatrix}3\cdot 1 + 1\cdot 0 + (-2)7\\3(-2) + 1\cdot 5 + (-2)0\\3\cdot 0 + 1\cdot 3 + (-2)11\end{bmatrix} = \begin{bmatrix}-11\\-1\\-19\end{bmatrix}$

3. (a) (3 points) Find all solutions to the matrix equation $A\mathbf{x} = \mathbf{b}$. Give your solution in parametric vector form.

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & -4 & 9 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$$

Solution: We row reduce the augmented matrix $\begin{bmatrix} -1 & 2 & -2 & -7 \\ 2 & -4 & 9 & 9 \end{bmatrix}$.

Add $2R1$ to $R2$: $\begin{bmatrix} -1 & 2 & -2 & -7 \\ 0 & 0 & 5 & -5 \end{bmatrix}$.

Multiply $R1$ by -1 and $R2$ by $1/5$: $\begin{bmatrix} 1 & -2 & 2 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Add $-2R2$ to $R1$: $\begin{bmatrix} 1 & -2 & 0 & 9 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

This is in RREF and the corresponding equations are now $x_1 - 2x_2 = 9$, $x_3 = -1$; So we have free variable $t = x_2$ and solutions are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t + 9 \\ t \\ -1 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \\ -1 \end{bmatrix}.$$

The RHS is the desired parametric form (with free variable t as the parameter).

(b) (1 point) Does the homogeneous equation $A\mathbf{x} = \mathbf{0}$ have any nonzero solutions? If so, what are they? (Hint: You can do this without computations by using your answer from part (a)!)

Solution: Our answer in part (a) tells us that $t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is a solution to the homogeneous system, for any $t \in \mathbf{R}$. This is because the parametric vector form of a solution will always have at most one vector with no free variable coefficient, and subtracting this gives the general form of the solution to the nonhomogeneous system.

Let's see why in this case: Set $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ 0 \\ -1 \end{bmatrix}$. We saw $t\mathbf{u} + \mathbf{v}$ is always a solution to $A\mathbf{x} = \mathbf{b}$ when A and $\mathbf{b}$ are as above, so in particular (setting $t = 0$) we have $A\mathbf{v} = \mathbf{b}$. Then for all $t \in \mathbf{R}$,

$$\mathbf{b} = A(t\mathbf{u} + \mathbf{v}) = A(t\mathbf{u}) + A\mathbf{v} = A(t\mathbf{u}) + \mathbf{b}.$$

Subtracting $\mathbf{b}$ from both sides, we see $A(t\mathbf{u}) = \mathbf{0}$, for all $t \in \mathbf{R}$.