Applied Linear Algebra
Instructor: Hachtman
Quiz 4 – 2/3/17

Name: ______________________________

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This quiz has 2 pages, a front and a back! No notes, calculators, phones etc. are permitted. **Show all your work.**

1. (a) (2 points) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Write the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution: We seek coefficients x_1, x_2 so that $x_1\mathbf{u} + x_2\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This is the same as solving the linear system with augmented matrix $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. So row reducing we get

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}.$$

So $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)\mathbf{u} + \frac{1}{3}\mathbf{v} = (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

(b) (2 points) Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation so that

$$T(\mathbf{u}) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad T(\mathbf{v}) = \begin{bmatrix} 12 \\ 9 \\ -3 \end{bmatrix}.$$

Use your answer from part (a) to write down the matrix representation of T.

Solution: We know the columns of the matrix of T are $T(\mathbf{e}_1) = T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $T(\mathbf{e}_2) = T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$. Since $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{e}_1$, we aready know the first column is $T(\mathbf{u}) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

We use the fact that T is linear for the second column, since

$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = T((-1)\mathbf{u}+\frac{1}{3}\mathbf{v}) = (-1)T(\mathbf{u})+\frac{1}{3}T(\mathbf{v}) = (-1)\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 12 \\ 9 \\ -3 \end{bmatrix} = \begin{bmatrix} -2+4 \\ -1+3 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

So the matrix of T is $\begin{bmatrix} 2 & 2 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}$.

2. (4 points) Suppose T is a linear transformation with matrix representation

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 8 & h \end{bmatrix}.$$

What values of h make T one-to-one? What values make T onto? Explain your answer.

Solution: We find the RREF of A, it's just $\begin{bmatrix} 1 & 4 & -1 \\ 0 & 0 & h+2 \end{bmatrix}$. We know that this is the matrix of a one-to-one linear transformation iff the columns are linearly independent. But they're not, and can never be, since there are three of them and these are vectors in $\mathbf{R}^2$–there would have to be a pivot in every column, but there are at most 2 pivots. We can also see directly that 4 times the first column is the second column. So no values of h make the transformation one-to-one.

We also know that the linear transformation is onto as long as the columns span $\mathbf{R}^2$, equivalently, as long as the matrix equation

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 0 & h+2 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

always has a solution. This is the case iff there is a pivot in every row (equivalently, there is no row of all zeroes in the RREF). So the equation always has a solution as long as $h + 2 \neq 0$, that is, $h \neq -2$. So T is onto whenever $h \neq -2$.

3. (2 points) Let

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 4 & 4 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}.$$

Determine which of the products AB and BA is defined, and compute the one that is.

Solution: AB is undefined. $BA = \begin{bmatrix} -3 & -4 & 1 & -2 \\ -6 & -8 & 2 & -4 \end{bmatrix}$.