Applied Linear Algebra
Instructor: Hachtman
Quiz 5 – 2/10/17

Name: ____________________

UIN: ____________________

This quiz has 2 pages, a front and a back! No notes, calculators, phones etc. are permitted. **Show all your work.**

1. Suppose $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -1 & -6 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(a) Solve the matrix equation $L\mathbf{y} = \begin{bmatrix} -6 \\ 10 \\ 4 \end{bmatrix}$.

Solution: This has augmented matrix $\begin{bmatrix} 1 & 0 & 0 & -6 \\ -2 & 1 & 0 & 10 \\ -1 & 0 & 1 & 4 \end{bmatrix}$. We row reduce:

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ -2 & 1 & 0 & 10 \\ -1 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

So we have the unique solution $\mathbf{y} = \begin{bmatrix} -6 \\ -2 \\ -2 \end{bmatrix}$.

(b) Use your answer from (a) to solve the matrix equation $A\mathbf{x} = \begin{bmatrix} -6 \\ 10 \\ 4 \end{bmatrix}$.

Solution: Since $A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = \mathbf{b}$, and we just found $\mathbf{y}$ so that $L\mathbf{y} = \mathbf{b}$, we must now solve the equation $U\mathbf{x} = \mathbf{y}$ using the $\mathbf{y}$ we found in the previous part. That is, we have to solve the matrix vector equation

$$U\mathbf{x} = \mathbf{y} \qquad \to \qquad \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -1 & -6 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -6 \\ -2 \\ -2 \end{bmatrix}$$

So we row-reduce the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 4 & -2 & -6 \\ 0 & -1 & -6 & -1 & -2 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 0 & -8 \\ 0 & -1 & -6 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -14 & 0 & -17 \\ 0 & 1 & 6 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So we have free variable x_3, and $x_1 = -17 + 14x_3$, $x_2 = 3 - 6x_3$, $x_4 = -1$. So solutions are of the form

$$\mathbf{x} = \begin{bmatrix} -17 + 14b \\ 3 - 6b \\ b \\ -1 \end{bmatrix} = \begin{bmatrix} -17 \\ 3 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 14 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \quad b \text{ any real number.}$$

2. (6 points) Let $A = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}$. Compute the following:

(a) A^{-1}

We row reduce the partitioned matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -5 & -1 \end{array}\right].$$

So the inverse is the right-hand matrix, $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ -1 & -5 & -1 \end{bmatrix}$.

(b) B^T

The transpose of B is just obtained by making the i, j-th entry equal to the j, i-th entry of B, or "flipping B along the diagonal":

$$B^T = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 4 \end{bmatrix}.$$

(c) $(B^{-1}A)^{-1}$

One could do this by inverting B, multiplying by A, then inverting that product. But this is a terrible idea. Instead, use the rule for inverses of products, $(XY)^{-1} = Y^{-1}X^{-1}$. Then $(B^{-1}A)^{-1} = A^{-1}(B^{-1})^{-1} = A^{-1}B$. Using our answer from (a),

$$\begin{aligned} A^{-1}B &= \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ -1 & -5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 3 + 5 \cdot 2 + 0 & 1 + 0 + 0 & 0 + 5 \cdot (-1) + 0 \\ 0 \cdot 3 + 1 \cdot 2 + 0 & 0 + 0 + 0 & 0 - 1 + 0 \\ -1 \cdot 3 + -5 \cdot 2 + 0 & -1 + 0 + 0 & 0 + (-1)(-5) - 4 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 1 & -5 \\ 2 & 0 & -1 \\ -13 & -1 & 1 \end{bmatrix} \end{aligned}$$