MATH 430
REVIEW FOR FINAL EXAM

1. Explain the difference (if any) between the sets:
   (a) $\varnothing$ and $\{\varnothing\}$;
   (b) $\{n \in \mathbb{N} \mid n \text{ is even}\}$ and $\{2n\}_{n\in\mathbb{N}}$;
   (c) $\{3, 7\}$ and $\langle 3, 7\rangle$;
   (d) $\{1, 3, 5\}$ and $\{5, 5, 1, 3\}$.
2. (a) Show $\{\leftrightarrow, \wedge, \vee\}$ is not a complete set of connectives.
   (b) Let $\sqcup$ denote "exclusive or"; that is, $\alpha \sqcup \beta$ is equivalent to $(\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta)$. Show $\{\neg, \sqcup\}$ is a complete set of connectives.
3. Consider a language $\mathcal{L}$ with a single binary relation symbol R. Show for $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$ that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, then $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent.
   Show the converse needn't hold, even if we assume $\mathfrak{A}$ and $\mathfrak{B}$ are both countable.
4. (a) Show that if Σ is a set of formulas in which the variable x doesn't appear free, then $\Sigma \models \psi$ implies $\Sigma \models \forall\psi$.
   (b) Give an example of a formula ψ so that $\psi \to \forall x\psi$ is not logically valid.
5. Consider the structure $\mathfrak{A} = (\mathbb{Z}; <)$ of integers with order.
   (a) Show the graph of the successor function, $\{\langle n, n+1\rangle \mid n \in \mathbb{Z}\}$, is a definable relation in $\mathfrak{A}$.
   (b) Similarly, show $\{\langle n, n+4\rangle \mid n \in \mathbb{Z}\}$ is definable in $\mathfrak{A}$.
   (c) Show that the only definable subsets of $\mathbb{Z}$ in $\mathfrak{A}$ are $\varnothing$ and $\mathbb{Z}$. (Hint: Given $R \subseteq \mathbb{Z}$ nontrivial, find an automorphism of $\mathfrak{A}$ that doesn't fix R.)
6. Consider the first order language with a single binary relation symbol, E. Let $\mathfrak{M} = (|\mathfrak{M}|; E^{\mathfrak{M}})$ be a structure so that $|\mathfrak{M}|$ is an infinite set, and $E^{\mathfrak{M}}$ is an equivalence relation on $\mathfrak{M}$ with precisely two classes, both of which are infinite.
   (a) Show $\operatorname{Th}\mathfrak{M}$ is $\aleph_0$-categorical, but not κ-categorical for some uncountable κ.
   (b) Is $\operatorname{Th}\mathfrak{M}$ finitely axiomatizable?
7. Consider a language $\mathcal{L}$ with a binary relation symbol E and unary relation symbols $R_1, \ldots, R_k$.
   (a) Give an example of a set Σ of $\mathcal{L}$-formulas so that for $\mathcal{L}$-structures $\mathfrak{A}$, we have $\mathfrak{A} \models \Sigma$ iff $(|\mathfrak{A}|; E^{\mathfrak{A}})$ is a graph and $\{\langle a, i\rangle \mid a \in |\mathfrak{A}| \text{ and } a \in R_i^{\mathfrak{A}}\}$ is a k-coloring of this graph.
   (b) Use the compactness theorem for first order logic to show that a graph is k-colorable if all of its finite subgraphs are. (Hint: You may need to adjoin constants for elements of the graph you are dealing with.)
8. We say a linear order $(L; <)$ is a *well-order* if any non-empty subset $A \subseteq L$ has a $<$-least element.
   Show being a well-order is not axiomatizable: Namely, show that whenever Σ is a set of $\{<\}$-sentences so that $(L; <) \models \Sigma$ for every well-order $(L; <)$, there is a model of Σ that is *not* a well-order. (This uses compactness—try adding infinitely many constants to the language.)
9. Show $\operatorname{Th}(\mathbb{Z}; <)$ is not $\aleph_0$-categorical.

10. (a) Define what it means for a theory T to admit quantifier elimination.
    (b) Let T be the theory of the structure $(|\mathfrak{A}|; P^{\mathfrak{A}})$, where $P^{\mathfrak{A}}$ is an infinite set with infinite complement in $|\mathfrak{A}|$. Show T admits quantifier elimination.
11. Show the theory of $\mathfrak{R}_{\text{sin}} = (\mathbb{R}; 0, 1, +, \cdot, \sin)$ is not decidable. (It is a theorem of Tarski that $(\mathbb{R}; 0, 1, +, \cdot)$ is decidable. The idea is that the function $\sin(x)$ lets you define $\mathfrak{N}$ inside $\mathfrak{R}_{\text{sin}}$; appeal to Gödel's incompleteness theorem.)
12. (Challenge problem.) Show that for any effectively enumerable set A of sentences with $A \subseteq \operatorname{Th}\mathfrak{N}$, there is a *complete* consistent theory $T \supseteq A$ so that

$$\#T = \{\#\sigma \mid \sigma \in T\}$$

is a definable set in $\mathfrak{N}$. (Note that by Gödel's incompleteness theorem, $\#T$ cannot be recursive; and by Tarski's theorem on non-definability of truth, $T \neq \operatorname{Th}\mathfrak{N}$.)