MATH 430
HOMEWORK 11 — DUE FRIDAY, APRIL 29

1. Show, for any finite $F \subseteq \mathrm{Th}\,\mathfrak{N}_S$, that $F \not\models \mathrm{Th}\,\mathfrak{N}_S$. (Recall here $\mathfrak{N}_S = (\mathbb{N}; 0, S)$. This uses compactness.)

2. Suppose $T$ is a theory so that for any formula $\varphi$ of the form $\exists x\theta$, with $\theta$ quantifier-free, there is a quantifier-free formula $\psi$ so that $T \models \varphi \leftrightarrow \psi$.
Show (by an induction on formulas) that $T$ admits quantifier elimination.

3. Show $\mathrm{Th}(\mathbb{Q}; <)$ admits quantifier elimination. (Note the language has equality.)

4. Recall a model $\mathfrak{M}$ of $\mathrm{Th}\,\mathfrak{N}$ is *nonstandard* if it is not isomorphic to $\mathfrak{N}$. In this problem you will show there are uncountably many *countable* nonstandard models of $\mathrm{Th}\,\mathfrak{N}$.

(a) Let $X \subseteq \mathbb{N}$ be arbitrary. Show there is a countable nonstandard model $\mathfrak{M}_X$ of $\mathrm{Th}\,\mathfrak{N}$ so that for some $a \in |\mathfrak{M}_X|$, we have $X = \{n \in \mathbb{N} \mid \mathfrak{M}_X \models (\exists x) x \cdot S^{p_n}(0) = a\}$. (Here $p_n$ is the $n$-th prime and $S^k(0)$ is the term for the $k$-th successor of 0.)
(Hint: Expand the language with a constant $c$ and use compactness.)

(b) For each $X \subseteq \mathbb{N}$, choose a model $\mathfrak{M}_X$ as in the previous part. Show the map $f : X \mapsto \mathfrak{M}_X$ is countable-to-one (that is, $f^{-1}(\mathfrak{M})$ is countable for all models $\mathfrak{M}$).

(c) Use this to argue that there are uncountably many countable models of $\mathrm{Th}\,\mathfrak{N}$ up to isomorphism.