Math 430: Formal Logic.

Solutions to Midterm #2

April 8, 2016

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1. (10 points) Let $\mathcal{L}$ be a first-order language, and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures.

   (a) (4 points) What does it mean for a function h to be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$?

   A homomorphism is a function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ such that h preserves all of the interpretations of the symbols of $\mathcal{L}$; more precisely,

   - if R is an n-ary relation symbol in $\mathcal{L}$, then $\langle a_1, \ldots, a_n \rangle \in R^{\mathfrak{A}} \iff \langle h(a_1), \ldots, h(a_n) \rangle \in R^{\mathfrak{B}}$, for all $a_1, \ldots, a_n \in |\mathfrak{A}|$;
   - if f is an n-ary function symbol in $\mathcal{L}$, then $h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$, for all $a_1, \ldots, a_n \in |\mathfrak{A}|$;
   - if c is a constant symbol in $\mathcal{L}$, then $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.

   (b) (6 points) Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, then they are elementarily equivalent.

   This is by induction on formula complexity. We show the stronger statement that whenever s is a variable assignment and ϕ is a formula, then $\mathfrak{A} \models \phi[s]$ iff $\mathfrak{B} \models \phi[h \circ s]$.

   Stepping stone: for any term t, $h(\bar{s}(t)) = \overline{h \circ s}(t)$.

   Well, supposing we have this, then for atomic formulas, we have

$$\begin{aligned}
\mathfrak{A}; s \models t_1 = t_2 &\iff \bar{s}(t_1) = \bar{s}(t_2) \\
&\iff h(\bar{s}(t_1)) = h(\bar{s}(t_2)) \text{ (by the assumption that } h \text{ is one-to-one)} \\
&\iff \overline{h \circ s}(t_1) = \overline{h \circ s}(t_2) \\
&\iff \mathfrak{B}; h \circ s \models t_1 = t_2.
\end{aligned}$$

   The same argument gives it for atomic formulas of the form $R(t_1, \ldots, t_n)$, using the assumption that h is a homomorphism.

   Now suppose inductively that we've proved it for formulas ψ, ϕ; we need to show it for $\psi \to \phi$, for $\neg\phi$, and for $\forall x \phi$.

   $\psi \to \phi$:

$$\begin{aligned}
\mathfrak{A}; s \models \psi \to \phi &\iff \mathfrak{A}; s \not\models \psi \text{ or } \mathfrak{A}; s \models \phi \\
&\iff \mathfrak{B}; h \circ s \not\models \psi \text{ or } \mathfrak{B}; h \circ s \models \phi \text{ (by inductive hypothesis)} \\
&\iff \mathfrak{B}; h \circ s \models \psi \to \phi.
\end{aligned}$$

   $\neg\phi$:

$$\begin{aligned}
\mathfrak{A}; s \models \neg\phi &\iff \mathfrak{A}; s \not\models \phi \\
&\iff \mathfrak{B}; h \circ s \not\models \phi \text{ (by inductive hypothesis)} \\
&\iff \mathfrak{B}; h \circ s \models \neg\phi.
\end{aligned}$$

   Finally, $\forall x \phi$:

$$\begin{aligned}
\mathfrak{A}; s \models \forall x \phi &\iff \text{for all } a \in |\mathfrak{A}|,\ \mathfrak{A}; s(x|a) \models \phi \\
&\iff \text{for all } a \in |\mathfrak{A}|,\ \mathfrak{B}; h \circ s(x|h(a)) \models \phi \text{ (by inductive hypothesis)} \\
&\iff \text{for all } b \in |\mathfrak{B}|,\ \mathfrak{B}; h \circ s(x|b) \models \phi \text{ (by assumption that h is surjective)} \\
&\iff \mathfrak{B}; h \circ s \models \forall x \phi.
\end{aligned}$$

2. (10 points) Suppose Γ is a set of sentences. Show that if $\Gamma \vdash \varphi$, then $\Gamma \vdash \forall x\varphi$. (If you use any theorems from class, you must state these precisely!)

There are two ways to do this. The first is probably easier, and uses the completeness and soundness theorems, which taken together, state: For any set of wffs Γ and wff ψ, $\Gamma \models \psi$ iff $\Gamma \vdash \psi$. In particular, it's sufficient to show that $\Gamma \models \phi$ implies $\Gamma \models \forall x\phi$.

So suppose $\Gamma \models \phi$. We want to conclude $\Gamma \models \forall x\phi$; so suppose $\mathfrak{A}; s$ is a structure and variable assignment so that $\mathfrak{A}; s \models \Gamma$. To show $\mathfrak{A}; s \models \forall x\phi$, it's enough to show that for every $a \in |\mathfrak{A}|$, $\mathfrak{A}; s(x|a) \models \phi$. Since Γ has no free variables, we have that $s(x|a)$ and s agree on all free variables appearing in Γ, so that by a theorem from class, $\mathfrak{A}; s(x|a) \models \Gamma$. Then by our assumption that $\Gamma \models \phi$, we have $\mathfrak{A}; s(x|a) \models \phi$, which is exactly what we needed.

We also could have proved this by induction on the length of the deduction of ϕ. Suppose ϕ is the last formula in the deduction $\langle \alpha_1, \ldots, \alpha_n \rangle$, and that inductively, we have the claim for all α_i with $i < n$. We have 3 cases:

1. $\phi \in \Lambda$. Then since Λ is closed under generalization, also $\forall x\phi \in \Lambda$.
2. $\phi \in \Gamma$. Then ϕ is a sentence, and in particular, x does not occur free in ϕ. "$\phi \to \forall x\phi$" is then a logical axiom, so we may apply modus ponens to deduce $\forall x\phi$ from Γ.
3. α_i is $\alpha_j \to \phi$ for some $i, j < n$. By our inductive hypothesis, $\forall x\alpha_i$, which is $\forall x(\alpha_j \to \phi)$, and $\forall x\alpha_j$, are both provable from Γ. Now

$$\forall x(\alpha_j \to \phi) \to \forall x\alpha_j \to \forall x\phi$$

   is a logical axiom. Applying modus ponens twice to this formula, we obtain a deduction of $\forall x\phi$.

3. (10 points) For each of the following sets, decide whether it is definable in the structure $(\mathbb{Z}; +)$; you must prove your answer.

   (a) $\{0\}$

   Yes, it's defined by the wff "$\forall v_2 v_1 + v_2 = v_2$".

   (b) $\{1\}$

   No. Consider the map $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(n) = -n$. This is a homomorphism, since $h(n+m) = -(n+m) = -n-m = h(n)+h(m)$; and is obviously 1-1 and onto. But $h(1) \neq 1$, so $\{1\}$ could not possibly be definable since definable sets are fixed by automorphisms.

   (c) $\{\langle x, y\rangle \in \mathbb{Z}^2 \mid x < y\}$

   The same automorphism as in part (b) witnesses that this set is not definable, since we have $x < y$ iff $h(y) < h(x)$, for all $x, y \in \mathbb{Z}$.

   (d) $\{\langle x, y\rangle \in \mathbb{Z}^2 \mid x = -y\}$

   Yes, this is definable: in brief, we can say "$v_1 + v_2 = 0$", or, in full, $(\exists v_3)((\forall v_2)(v_3 + v_2 = v_2) \wedge v_1 + v_2 = v_3)$.

   (e) $\{\langle x, y, z\rangle \in \mathbb{Z}^3 \mid x \cdot y = z\}$

   No, not. And the same automorphism works: $\langle 1, 1, 1\rangle$ is in this set, but its pointwise image under h, $\langle -1, -1, -1\rangle$, is not.

4. (10 points) Let $\mathcal{L}$ be the language with equality plus a single binary relation symbol, E. We say a graph $\mathcal{G} = (X; E^{\mathcal{G}})$ is *connected* if whenever $x, y \in X$ are distinct, there is a finite sequence $\langle x_1, x_2, \ldots, x_n\rangle$ so that $x_1 = x$, $x_n = y$, and $\langle x_i, x_{i+1}\rangle \in E^{\mathcal{G}}$ for all $1 \leq i < n$.

   (a) (4 points) Write down an $\mathcal{L}$-sentence that is not satisfied by any connected graph.

   For example: "$\exists v_1 \exists v_2 (v_1 \neq v_2 \wedge \forall v_3(\neg E v_1 v_3))$" works: This says that there are at least two vertices, but there is some vertex who has no edge-neighbor.

   (b) (6 points) Show there is no set Σ of $\mathcal{L}$-sentences such that: $\mathcal{G} \models \Sigma$ iff $\mathcal{G}$ is a connected graph.

   Expand the language by two constant symbols c, d and let θ_n be the sentence in the expanded language that says c, d are not connected by any path of length n; this could be written

$$\forall v_1 \forall v_2 \forall v_3 \ldots \forall v_n (\neg{=}v_1 c) \vee (\neg E v_1 v_2) \vee (\neg E v_2 v_3) \vee \cdots \vee (\neg E v_{n-1} v_n) \vee (\neg{=}v_n d).$$

   Now suppose towards a contradiction that Σ is a set of sentences satisfied by precisely the connected graphs. We claim the theory $\Sigma \cup \{\theta_n\}_{n\in\mathbb{N}}$ is satisfiable. By compactness, it is enough to show that every finite subset is satisfiable; indeed, we show $\Sigma \cup \{\theta_n\}_{n\leq N}$ is satisfiable, for each natural number N. We let $\mathcal{G}_N$ be the graph with $N+1$ vertices, $x_1, \ldots, x_{N+1}$, so that $\langle x_i, x_{i+1}\rangle \in E^{\mathcal{G}}$ for all $i \leq N$; we interpret $c^{\mathcal{G}_N} = x_1$ and $d^{\mathcal{G}_N} = x_{N+1}$. Since the shortest path joining x_1 and x_{N+1} has length $N+1$, this graph must model θ_i for all $i \leq N$, and since the graph is connected, it must satisfy Σ.

   Now by compactness we get a model $\mathcal{G}^*$ of $\Sigma \cup \{\theta_n\}_{n\in\omega}$. Regarding this as purely a graph (by forgetting about the interpretations of c, d), this must be a connected graph, since it satisfies Σ. Now $c^{\mathcal{G}^*}$ and $d^{\mathcal{G}^*}$ are vertices in $|\mathcal{G}^*|$, so must be joined by a path; say this path has length k. But this contradicts the fact that $\mathcal{G}^*$ satisfies θ_k. This contradiction finishes the proof.

5. (10 points) Let E be the set of even integers, $\{2n \mid n \in \mathbb{Z}\}$. Show $(\mathbb{R}; E)$ and $(\mathbb{Z}; E)$, regarded as structures in the language with a single unary predicate P, are elementarily equivalent.

   By the downward Lowenheim Skolem theorem, there is a structure $\mathfrak{C}$ that is countable, and so that $\mathfrak{C}$ and $(\mathbb{R}; E)$ are elementarily equivalent.

   Now for each n, $(\mathbb{R}; E)$ and $(\mathbb{Z}; E)$ both satisfy the sentence stating: "There are at least n distinct objects x so that $P(x)$ holds." As well as: "There are at least n distinct objects so that $P(x)$ fails." In particular, whatever $\mathfrak{C}$ is, we have that $|\mathfrak{C}|$ is a countably infinite set, and both $P^{\mathfrak{C}}$ and $|\mathfrak{C}| \setminus P^{\mathfrak{C}}$ are countably infinite.

   Since E and $\mathbb{Z} \setminus E$ are both countably infinite, we may then fix bijections $h_1 : P^{\mathfrak{C}} \to E$ and $h_2 : |\mathfrak{C}| \setminus P^{\mathfrak{C}} \to \mathbb{Z} \setminus E$. Then $h := h_1 \cup h_2$ is a bijection between $|\mathfrak{C}|$ and $\mathbb{Z}$, and what's more, it is a homomorphism of $\mathcal{L}$-structures $\mathfrak{C}$ and $(\mathbb{Z}; E)$.

   So $\mathfrak{C}$ and $(\mathbb{Z}; E)$ are isomorphic, hence by problem 1, are elementarily equivalent. By design, $\mathfrak{C}$ and $(\mathbb{R}; E)$ are elementarily equivalent. So by transitivity of elementarily equivalence, we have that $(\mathbb{R}; E)$ and $(\mathbb{Z}; E)$ are elementarily equivalent.