MATH 430: FORMAL LOGIC
SPRING 2018
REVIEW PROBLEMS 1

1. Suppose $f : A \to B$ is a function. Show the following:

(a) If f is one-to-one then $A \sim f[A]$.
(b) If f is one-to-one then $f \restriction X$ is one-to-one, for all $X \subseteq A$.
(c) f is one-to-one if and only if $f[X \cap Y] = f[X] \cap f[Y]$ for all $X, Y \subseteq A$.

2. Suppose $f : A \to B$ is a function. For each of the following sets, give a definition of it in set-builder notation, i.e. $\{v \in Y \mid \varphi(v)\}$, where φ uses only symbols of the first order language of set theory and possibly the symbols $\langle \cdot \rangle$ for ordered tuples.

(a) range(f)
(b) codomain(f)
(c) $f[B \setminus A]$
(d) $f^{-1}[A \cap B]$
(f) The set of *fixed points* of f, where x is a fixed point of f iff $f(x) = x$
(e) $\bigcap\{X \subseteq A \mid f \restriction X \text{ is one-to-one}\}$

3. The *successor operation* S is defined by letting $S(x) = x \cup \{x\}$, for all sets x. Show the function $S : V \to V$ is injective (where V is the class of all sets).

4. Define a relation $\equiv$ on $\mathbb{N} \times \mathbb{N}$ by letting

$$\langle a, b\rangle \equiv \langle c, d\rangle \iff a + d = c + b.$$

(a) Show $\equiv$ is an equivalence relation.
(b) Let $\mathcal{Z}$ be the collection of $\equiv$-equivalence classes,

$$\mathcal{Z} = \{[\langle a, b\rangle]_\equiv \mid a, b \in \mathbb{N}\}.$$

Define a binary operation $\oplus$ on $\mathcal{Z}$ by letting $[\langle a, b\rangle]_\equiv \oplus [\langle c, d\rangle]_\equiv = [\langle a+c, b+d\rangle]_\equiv$. Show this operation is well-defined (that is, for $s, t \in \mathcal{Z}$, the output $s \oplus t$ does not depend on the choice of representatives $\langle a, b\rangle$ and $\langle c, d\rangle$ for the classes s, t).
(c) Show $\oplus$ is commutative and associative.
(d) Show there is an additive identity $0^{\mathcal{Z}}$, that is, there is a unique $0^{\mathcal{Z}} \in \mathcal{Z}$ so that $s \oplus 0^{\mathcal{Z}} = s$ for all $s \in \mathcal{Z}$.
(d) Show the existence of additive inverses: For all $s \in \mathcal{Z}$ there is a unique $t \in \mathcal{Z}$ with $s \oplus t = 0^{\mathcal{Z}}$.
(e) How should we define multiplication $\otimes$ in $\mathcal{Z}$? What is the multiplicative identity $1^{\mathcal{Z}}$?