MATH 430: FORMAL LOGIC
SPRING 2018
SOLUTIONS TO HOMEWORK 1

1. Write the truth table and find a disjunctive normal form.

(a) $P \vee Q \wedge \neg(Q \to P)$

Note our convention that "$\wedge$" has priority over "$\vee$" dictates that this be parsed as $P \vee (Q \wedge \neg(Q \to P))$.

P	Q	$Q \to P$	$\neg(Q \to P)$	$Q \wedge \neg(Q \to P)$	$P \vee (Q \wedge \neg(Q \to P))$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	T	F	F	F

There are three "T" rows, each one corresponding to a disjunct in the disjunctive normal form $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$.

(b) $(P \vee Q) \leftrightarrow \neg(Q \wedge R)$

P	Q	R	$P \vee Q$	$Q \wedge R$	$\neg(Q \wedge R)$	$(P \vee Q) \leftrightarrow \neg(Q \wedge R)$
T	T	T	T	T	F	F
T	T	F	T	F	T	T
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	F	F
F	T	F	T	F	T	T
F	F	T	F	F	T	F
F	F	F	F	F	T	F

One d.n.f. is $(P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R)$. Again note how each disjunct corresponds to a row of the truth table where the statement is assigned "T".

2. We have to find formulas in the language of arithmetic, $\mathcal{L}_{\text{arith}} = \{\bar{0}, S, +, \cdot, <\}$; so no other non-logical symbols can appear in the formula we write, and quantifiers automatically have scope $\mathbb{N}$.

(a) n is prime iff whenever we have $x \cdot y = n$ for naturals x, y, then one of x and y is equal to n. So one way of expressing "n is prime" is

$$\forall v_1 \forall v_2 (v_1 \cdot v_2 = n \to (v_1 = n \vee v_2 = n)).$$

(b) Note here that saying "there are infinitely many m such that..." in this context is the same as saying "there are unboundedly many m ...", that is, "for all n there is an m greater than n such that..."; so for any property $Q(n)$ of naturals, "there are infinitely many n such that $Q(n)$" is expressed by

$$\forall v_1 \exists v_2 (v_1 < v_2 \wedge Q(v_2)).$$

And (letting $P(n)$ be the formula we wrote in part (a)) n is the lesser in a pair of twin primes iff $P(n) \wedge P(S(S(n)))$ (recall S is the successor operation, $S(n) = n + 1$). So the twin prime conjecture is expressed by the sentence

$$\forall v_1 \exists v_2 (v_1 < v_2 \wedge (P(v_2) \wedge P(S(S(v_2))))).$$

By the way, our book defines the language of arithmetic as just $\{\bar{0}, S, +, \cdot\}$ (without $<$). Since $v_1 < v_2$ is equivalent to $\exists v_3 (\neg v_3 = \bar{0}) \wedge v_2 = v_1 + v_3$ in the natural numbers, we could rewrite this formula without using $<$ as

$$\forall v_1 \exists v_2 (\exists v_3 ((\neg v_3 = 0) \wedge v_2 = v_1 + v_3) \wedge (P(v_2) \wedge P(S(S(v_2))))).$$

3. Say our language is $\{R\}$ with R binary. Once again, the point is that we have to find a sentence whose only non-logical symbol is R. The structure $(\mathbb{N}, \sqsubset)$ is the strict linear order that looks like

$$0 < 2 < 4 < 6 < \cdots < 1 < 3 < 5 < \cdots$$

But we have no constants or function symbols, so talking about "0" or "1" or "even" makes no sense in this language. We do see, however, that in $(\mathbb{N}, \sqsubset)$, the element 1 has no immediate predecessor. This is also true of 0 here, and of 0 in $(\mathbb{N}, <)$; but in $(\mathbb{N}, \sqsubset)$, 1 is an object that has no immediate predecessor *and* is *not* minimal. So the sentence asserting the existence of such an object

$$\exists v_1 (\exists v_2 R(v_2 v_1) \wedge \forall v_2 (R(v_2 v_1) \to \exists v_3 (R(v_2 v_3) \wedge R(v_3 v_1))))$$

is modeled by $(\mathbb{N}, \sqsubset)$ and not by $(\mathbb{N}, <)$.