MATH 430: FORMAL LOGIC
SPRING 2018
HOMEWORK 2 SOLUTIONS

1. Let T be a first order theory and φ be a formula.

(a) Prove that $T \vdash \neg\varphi$ if and only if $T \cup \{\varphi\}$ is inconsistent.
(b) Prove that if T is inconsistent, then $T \vdash \psi$ for *all* formulas ψ.

(a) First suppose $T \vdash \neg\varphi$. If s is a deduction of $\neg\varphi$ from T, then we can extend s to a proof from $T \cup \{\varphi\}$, by the steps:

- $\neg\varphi \to (\varphi \to \varphi \wedge \neg\varphi)$ (tautology),
- $\varphi \to \varphi \wedge \neg\varphi$ (modus ponens from previous two lines),
- φ (assumption),
- $\varphi \wedge \neg\varphi$ (modus ponens from previous two lines).

This shows $T \cup \{\varphi\}$ is inconsistent.

Suppose $T \cup \{\varphi\}$ is inconsistent. By definition, $T \cup \{\varphi\} \vdash \psi \wedge \neg\psi$ for some formula ψ. By the deduction theorem, we have $T \vdash (\varphi \to (\psi \wedge \neg\psi))$. Let s be a deduction witnessing this. Now

$$(\varphi \to (\psi \wedge \neg\psi)) \to \neg\varphi$$

is a tautology. Therefore we can add two steps to the deduction s: the displayed tautology, followed by $\neg\varphi$ (by an application of Modus Ponens). This is then a deduction of $\neg\varphi$ from T.

(b) Notice $(\varphi \wedge \neg\varphi) \to \psi$ is a tautology for any choice of formulas φ, ψ. Using this it's easy to get a proof of ψ from an inconsistent theory, for any formula ψ.

2. Write down a deduction witnessing $\emptyset \vdash \forall x\varphi \to \exists x\varphi$.

(1) $(\forall x\varphi \to \varphi) \to ((\varphi \to \exists x\varphi) \to (\forall x\varphi \to \exists x\varphi))$ (Tautology)
(2) $\forall x\varphi \to \varphi$ (Axiom 2)
(3) $(\varphi \to \exists x\varphi) \to (\forall x\varphi \to \exists x\varphi)$ (Modus ponens, lines 1 and 2)
(4) $(\forall x\neg\varphi \to \neg\varphi) \to (\varphi \to \neg\forall x\neg\varphi)$ (Tautology)
(5) $\forall x\neg\varphi \to \neg\varphi$ (Axiom 2)
(6) $\varphi \to \neg\forall x\neg\varphi$ (Modus ponens, lines 4 and 5)
(7) $(\varphi \to \neg\forall x\neg\varphi) \to ((\exists x\varphi \leftrightarrow \neg\forall x\neg\varphi) \to (\varphi \to \exists x\varphi))$ (Tautology)
(8) $(\exists x\varphi \leftrightarrow \neg\forall x\neg\varphi) \to (\varphi \to \exists x\varphi)$ (Modus ponens, lines 6 and 7)
(9) $\exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$ (Axiom 5)
(10) $\varphi \to \exists x\varphi$ (Modus ponens, lines 8 and 9)
(11) $\forall x\varphi \to \exists x\varphi$ (Modus ponens, lines 3 and 10)

The most subtle point in this deduction is the use of Axiom 2, applied to each of the formulas φ, $\neg\varphi$. Recall Axiom 2 says $\forall x P(x) \to P(t)$ is an axiom, whenever t is a term such that *no free variable in t is bound by a quantifier of $P(t)$*; recall $P(t)$ is obtained by replacing all free occurrences of x in $P(x)$ by t. In line 2 of the above deduction, we are taking P to be φ, and the term t to just be x. And x is always a term t with the property just mentioned—we're just replacing *free* occurrences of x by $t = x$, so no such "replacement" results in x being bound!

You should convince yourself, also, that all of the lines listed as tautologies really are: For example, the first line has the form $(P \to Q) \to ((Q \to R) \to (P \to R))$.

3. Show that no one of the following sentences is logically implied by the other two.

(a) $\forall x \forall y \forall z (P(x, y) \to (P(y, z) \to P(x, z)))$
(b) $\forall x \forall y (P(x, y) \to (P(y, x) \to x = y))$
(c) $\forall x \exists y P(x, y) \to \exists y \forall x P(x, y)$

For each part we give a structure that does not model that part, but does satisfy the other two.

(a) $(\{0, 1, 2\}, R)$, where $\langle i, j\rangle \in R$ iff $j = i + 1$.
(b) $(\{0, 1\}, \{0, 1\} \times \{0, 1\})$.
(c) $(\mathbb{N}, \leq)$.

4. Prove $T \models \varphi \wedge \neg\varphi$ if and only if T is unsatisfiable.

For the forward direction, we prove the contrapositive. Suppose T is satisfiable, so there are a structure $\mathcal{A}$ and assignment $s : V \to A$ so that $\mathcal{A} \models T[s]$. It's not too hard to see $\mathcal{A} \not\models (\varphi \wedge \neg\varphi)[s]$, since by the inductive definition of satisfaction, $\mathcal{A} \models (\neg\varphi)[s]$ iff $\mathcal{A} \not\models \varphi[s]$. So $\mathcal{A}, s$ witness that $T \not\models \varphi \wedge \neg\varphi$.

Conversely, suppose T is unsatisfiable. Then every model of T is a model of $\varphi \wedge \neg\varphi$, since there *aren't any* models of T—that is, we have $T \models \varphi \wedge \neg\varphi$ vacuously.