MATH 430: FORMAL LOGIC
SPRING 2018
HOMEWORK 3

Due Friday, February 23.

1. Suppose $f : A \to B$ is injective *and that A is non-empty*. Show there is a surjection $g : B \to A$.

2. Show $\mathbb{R} \sim 2^{\mathbb{N}}$.

3. Recall the language of rings is $\mathcal{L}_{\text{rings}} = \{0, 1, +, \cdot\}$. For A a set, we may define an $\mathcal{L}_{\text{rings}}$-structure $\mathcal{A} = (\mathcal{P}(A), \emptyset, A, \cup, \cap)$; so for example, $0^{\mathcal{A}} = \emptyset$, and $+^{\mathcal{A}} = \cup$.

Which of the ring and field axioms are modeled by this structure? (Don't forget to consider the case $A = \emptyset$!) For those axioms that fail, explain why.

4. We say a relation R' **extends** a relation R if for all a, b, $\langle a, b\rangle \in R$ implies $\langle a, b\rangle \in R'$; equivalently, $R \subseteq R'$.

Suppose $\mathcal{B} = (B, \leq)$ is a partial order with B finite. Show there is a linear order $(B, \trianglelefteq)$ such that $\trianglelefteq$ extends $\leq$.

5. Explicitly define (without appealing to the Cantor-Schröder-Bernstein Theorem) a bijection $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

6. Let C be a set. A family $\mathcal{D}$ of subsets of C is a **partition of** C if $C = \bigcup \mathcal{D}$ and $a \cap b =$ whenever $a, b \in \mathcal{D}$ are distinct. Show

$$\{\mathcal{D} \mid \mathcal{D} \text{ is a partition of } C\} \sim \{R \mid R \text{ is an equivalence relation on } C\}.$$

7. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure for some first order language $\mathcal{L}$ and $f, g : V \to A$ are variable assignments. Prove by an induction on the complexity of terms t that if $f(v_i) = g(v_i)$ for all v_i appearing in t, then $\hat{f}(t) = \hat{g}(t)$.

8. (Extra credit.)

(a) Suppose $\mathcal{C}$ is a partition of $\mathbb{R}$ such that every element of $\mathcal{C}$ is a half-closed interval $(a,\ b\,]$ with $a < b$ in $\mathbb{R}$. Show $\mathcal{C}$ is countable.

(b) Show $\mathbb{R} \sim \{f \in \mathbb{R}^{\mathbb{R}} \mid f : \mathbb{R} \to \mathbb{R} \text{ is continuous}\}$.