MATH 430: FORMAL LOGIC
SPRING 2018
HOMEWORK 3 SOLUTIONS

1. We are given an injection $f : A \to B$, and are told that A is non-empty. Since A is non-empty, we can fix $a_0 \in A$. Now let, for $y \in B$,

$$g(y) = \begin{cases} x & \text{if } x \text{ is unique such that } f(x) = y, \\ a_0 & \text{if no such } x \text{ exists.} \end{cases}$$

This is indeed a function with domain B: if $y \in \text{range}(f)$, then there is some $x \in A$ with $f(x) = y$; since f is injective, this x must be unique. So we have $g(y)$ is defined, and equal to x. Otherwise, $y \notin \text{range}(f)$, and so $g(y) = a_0$.

And $g : B \to A$ is surjective: since if $x \in A$, then $g(f(x)) = x$.

2. We must show $\mathbb{R} \sim 2^{\mathbb{N}}$. By Cantor-Schröder-Bernstein, it's sufficient to show $\mathbb{R} \preceq 2^{\mathbb{N}}$ and $2^{\mathbb{N}} \preceq \mathbb{R}$, that is, that there are injections in both directions.

This problem is perhaps easiest if we regard real numbers as infinite decimal expansions. So for example, if we let $G : 2^{\mathbb{N}} \to \mathbb{R}$ take a binary sequence $\langle n_i \rangle_{i \in \mathbb{N}}$ to the real whose decimal expansion is that sequence,

$$G(\langle n_i \rangle_{i \in \mathbb{N}}) = 0.n_0 n_1 n_2 \cdots = \sum_{i=1}^{\infty} \frac{n_i}{10^i}$$

then G is injective (why?).

One possible injection $H : \mathbb{R} \to 2^{\mathbb{N}}$ takes a real input $r = r_0.r_1 r_2 r_3 \ldots$ in decimal expansion, and outputs a binary sequence that codes this, beginning with r_0 1's, followed by a 0, then r_1 1's, followed by a 0, then r_2 1's, and so on. So for example,

$$4.01301 \mapsto 1111001011100100000000\ldots$$

3. If $A \neq \emptyset$, then the axiom asserting existence of additive inverse fails, since e.g. $A \cup X = A \neq 0^{\mathcal{A}} = \emptyset$ for any $X \subseteq A$.

Similarly, multiplicative inverse fails if $A \neq \emptyset$, since e.g. $\emptyset \cap X = \emptyset \neq 1^{\mathcal{A}} = A$, for all $X \subseteq A$.

Now if $A = \emptyset$, then $\mathcal{P}(A)$ is the 1-element set $\{\emptyset\}$. Note this satisfies all of the field axioms, except for the requirement $0 \neq 1$.

5. One idea is to "interleave" digits of the binary expansion. For example,

$$m(\langle 0.14512, 0.\overline{331} \rangle) = 0.134351132301\overline{030301}$$

(Note we should decide whether to use the non-repeating decimal expansion, e.g. 0.301 versus 0.3009999..., for this to be well-defined). This gets messy if we consider negative numbers; so one work-around would be to restrict this "mixing" function m to the open interval $(0,1) \times (0,1)$, and compose with a bijection $g : \mathbb{R} \to (0,1)$; for example, $g(x) = \tan^{-1}(x)/\pi + 1/2$. Then $f(x,y) = g^{-1}(m(g(x), g(y)))$ defines a bijection $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

6. The idea is that the set of equivalence classes of an equivalence relation determine a unique partition, and vice versa. Here is a proof.

Let $\mathfrak{P}$ denote the set of partitions of C, and $\mathfrak{E}$ the set of equivalence relations on C. We claim that $F : \mathfrak{P} \to \mathfrak{E}$ defined by letting

$$F(\mathcal{D}) = \{\langle x, y\rangle \in C \times C \mid (\exists B \in \mathcal{D}) x \in B \wedge y \in B\}$$

is a bijection.

Clearly $F(\mathcal{D})$ is uniquely defined for each $\mathcal{D}$, but we must check it belongs to $\mathfrak{E}$; that is, that $F(\mathcal{D})$ is an equivalence relation.

- Reflexive: Every $x \in C$ belongs to some piece B of the partition. So $\langle x, x\rangle \in F(\mathcal{D})$ for all $x \in C$.
- Symmetric: Suppose $\langle x, y\rangle \in F(\mathcal{D})$. Say this is witnessed by some $B \in \mathcal{D}$. Since both $y \in B$ and $x \in B$, we have $\langle y, x\rangle \in F(\mathcal{D})$.
- Transitive: Suppose $\langle x, y\rangle, \langle y, z\rangle \in F(\mathcal{D})$. Say this is witnessed by B_1 and B_2 respectively. Then $y \in B_1 \cap B_2$; since $B_1 \cap B_2 \neq \emptyset$, we must have $B_1 = B_2$ by the definition of partition. Then $x, z \in B_1 = B_2$, so $\langle x, z\rangle \in F(\mathcal{D})$ as needed.

So we have our function $F : \mathfrak{P} \to \mathfrak{E}$.

We have to show F is one-to-one. Suppose $\mathcal{D}_1$, $\mathcal{D}_2$ are distinct partitions. Without loss of generality, suppose there is some $B_1 \subseteq C$ that is in $\mathcal{D}_1$ but not in $\mathcal{D}_2$. Let $x \in B_1$. Since $\mathcal{D}_2$ is a partition we have $C = \bigcup \mathcal{D}_2$ and so $x \in B_2$ for some $B_2 \in \mathcal{D}_2$. Since $B_1 \notin \mathcal{D}_2$, we have $B_1 \neq B_2$. So there must be some $y \neq x$ belonging to one set and not the other. Then we have $\langle x, y\rangle$ belongs to one of $F(\mathcal{D}_1), F(\mathcal{D}_2)$, and not the other. In particular these values are distinct, which shows F is one-to-one.

Finally, F is onto: If $R \in \mathfrak{E}$, then let

$$\mathcal{D}_R = \{[x]_R \mid x \in C\}.$$

This is just the set of R-equivalence classes. Check that $\mathcal{D}_R$ is a partition. And then, that $F(\mathcal{D}_R) = R$. Thus F is onto.

We have that F is a bijection, so that $\mathfrak{P} \sim \mathfrak{E}$.