MATH 430: FORMAL LOGIC
SPRING 2018
HOMEWORK 4

Due Friday, March 2.

1. The *successor operation* S is defined by letting $S(x) = x \cup \{x\}$, for all sets x. Show the function $S : V \to V$ is injective (where V is the class of all sets).

2. Consider a binary relation defined on triples of numbers,

$$\langle a, b, c\rangle \sim \langle d, e, f\rangle \iff af + ce = cd + bf.$$

(a) Show $\sim$ is an equivalence relation on $\mathbb{N}\times\mathbb{N}\times(\mathbb{N}\setminus\{0\})$, but *not* on $\mathbb{N}^3 = \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

(b) Consider the collection of $\sim$-equivalence classes,

$$\mathcal{Q} := \{[\langle a, b, c\rangle]_\sim \mid a, b, c \in \mathbb{N}, c \neq 0\}.$$

Define an operation $\oplus$ on $\mathcal{Q}$ by letting $\langle a, b, c\rangle \oplus \langle d, e, f\rangle = \langle af + cd, bf + ce, cf\rangle$. Show this operation is well-defined.

(c) Show there is an additive identity $0^{\mathcal{Q}}$: that is, some $0^{\mathcal{Q}}$ so that $0^{\mathcal{Q}} \oplus r = r \oplus 0^{\mathcal{Q}} = r$ for all $r \in \mathcal{Q}$.

(d) Show $\oplus$ is commutative and associative on $\mathcal{Q}$.

(e) Define an appropriate multiplication operation $\otimes$ on $\mathcal{Q}$. What element of $\mathcal{Q}$ is the multiplicative identity? Show any $r \in \mathcal{Q}$ with $r \neq 0^{\mathcal{Q}}$ has a multiplicative inverse.

3. Let $(A, \leq_A)$, $(B, \leq_B)$ be well-orders. Show the product $(A \times B, \leq_{\text{Lex}})$, where $\leq_{\text{Lex}}$ is the lexicographic order of $\leq_A$ and $\leq_B$, is a well-order.

4. Consider the structure $(\mathbb{N}^{\mathbb{N}}, \leq)$, where $f \leq g$ iff $f = g$ or if $n \in \mathbb{N}$ is least such that $f(n) \neq g(n)$, we have $f(n) < g(n)$. Show this is a linear order, but not a well-order.

5. Suppose $f : A \to B$ is surjective. Show there is an injection $g : B \to A$. Be sure to state where in your proof you use the Axiom of Choice.

6 (Extra credit)**.** For sets A, let $\mathcal{P}_{\text{well}}(A) = \{x \subseteq A \mid x \text{ can be well-ordered}\}$. Assuming Choice, $\mathcal{P}_{\text{well}}(A) = \mathcal{P}(A)$; nonetheless, show without using Choice that there can be no injection $f : \mathcal{P}_{\text{well}}(A) \to A$.