MATH 430: FORMAL LOGIC
SPRING 2018
SOLUTIONS TO HOMEWORK 4

1. The *successor operation* S is defined by letting $S(x) = x \cup \{x\}$, for all sets x. Show the function $S : V \to V$ is injective (where V is the class of all sets).
Solution. Suppose $S(x) = S(y)$. To show injectivity of S, we need to show $x = y$. Note that by definition, $z \in S(x)$ iff $z = x$ or $z \in x$. Then we have for all z,

$$(z \in x \text{ or } z = x) \text{ if and only if } (z \in y \text{ or } z = y).$$

Now $x \in S(x)$ and $y \in S(y)$. So we have $x = y$ or $x \in y$, and $y = x$ or $x \in y$.

So if $x \neq y$, we must have $x \in y$ and $y \in x$. But then $\{x, y\}$ has no $\in$-minimal element, contradicting the Axiom of Regularity (Foundation).

3. Let $(A, \leq_A)$, $(B, \leq_B)$ be well-orders. Show the product $(A \times B, \leq_{\text{Lex}})$, where $\leq_{\text{Lex}}$ is the lexicographic order of $\leq_A$ and $\leq_B$, is a well-order.
Solution. We just show it's well-founded. Suppose $Y \subseteq A \times B$ is non-empty. Then

$$A_0 = \{a \in A \mid \langle a, b\rangle \in Y \text{ for some } b \in B\}$$

is also non-empty. Since Y is a well-order, we may take a_0 to be $\leq_A$-minimal in A_0. Then let

$$B_0 = \{b \in B \mid \langle a_0, b\rangle \in Y\}.$$

By the fact that $\leq_B$ is a well-order, we take b_0 to be $\leq_B$-minimal in B_0.

Now clearly $\langle a_0, b_0\rangle \in Y$. We claim it is $\leq_{\text{Lex}}$-minimal in Y. For suppose $\langle a_1, b_1\rangle <_{\text{Lex}} \langle a_0, b_0\rangle$. Then either $a_1 <_A a_0$, or $a_1 = a_0$ and $b_1 <_B b_0$.

If $a_1 <_A a_0$ then $a_1 \notin A_0$ by minimality, so there is no $b \in B$ so that $\langle a_1, b\rangle \in Y$; in particular $\langle a_1, b_1\rangle \notin Y$.

If $a_1 = a_0$ then $b_1 <_B b_0$. Then $b_1 \notin B_0$, again by minimality of b_0. That is, $\langle a_0, b_1\rangle \notin Y$. Since $a_0 = a_1$, we have $\langle a_1, b_1\rangle \notin Y$.

We have shown any point strictly $<_{\text{Lex}}$-below $\langle a_0, b_0\rangle$ does not belong to Y; that is, $\langle a_0, b_0\rangle$ is the $\leq_{\text{Lex}}$-least element of Y, as needed.

4. Consider the structure $(\mathbb{N}^{\mathbb{N}}, \leq)$, where $f \leq g$ iff $f = g$ or if $n \in \mathbb{N}$ is least such that $f(n) \neq g(n)$, we have $f(n) < g(n)$. Show this is a linear order, but not a well-order.
Solution. We'll just show transitivity. Suppose $f \leq g$ and $g \leq h$. If either of these is an equality, then $f \leq h$ easily. So there are n_1 and n_2, each minimal so that $f(n_1) \neq g(n_1)$ and $g(n_2) \neq h(n_2)$, respectively. Then by assumption $f(n_1) < g(n_1)$ and $g(n_2) < h(n_2)$.

We now have three cases:

(1) $n_1 < n_2$. Then $g(n_1) = h(n_1)$, and so $f(n_1) < g(n_1) = h(n_1)$, and also (by minimality of n_1, n_2 and transitivity of $=$) $f(k) = h(k)$ for $k < n_1$.

(2) $n_1 = n_2$. Then $f(n_1) < g(n_1) = g(n_2) < h(n_2)$, so we have (again that $f(k) = g(k) = h(k)$ for $k < n_1 = n_2$ and thus) $f \leq h$.

(3) $n_1 > n_2$. Then $f(n_2) = g(n_2) < h(n_2)$; as before, $f \leq h$.

To see that $\leq$ is not a well-order, we need a strictly decreasing sequence $\langle f_k \rangle$ of functions in $\mathbb{N}^{\mathbb{N}}$. We can for instance put

$$f_k(n) = \begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

That is,

$$\begin{aligned} f_0 &= \langle 1, 1, 1, 1 \ldots \rangle \\ f_1 &= \langle 0, 1, 1, 1 \ldots \rangle \\ f_2 &= \langle 0, 0, 1, 1 \ldots \rangle \\ f_3 &= \langle 0, 0, 0, 1 \ldots \rangle \end{aligned}$$

You should convince yourself that these are strictly decreasing.

5. Suppose $f : A \to B$ is surjective. Show there is an injection $g : B \to A$. Be sure to state where in your proof you use the Axiom of Choice.

Solution. Let

$$D = \{f^{-1}(b) \mid b \in B\}.$$

Here $f^{-1}(b)$ denotes the inverse image of $b \in B$, that is, $f^{-1}(b) = \{a \in A \mid f(a) = b\}$.

Note D is a collection of non-empty pairwise disjoint subsets of A. (Non-empty because f is surjective, which means precisely that $f^{-1}(b) \neq \emptyset$ for all $b \in B$. Pairwise disjoint because f is a function.)

Now by the Axiom of Choice, there is a fiber F through D, that is, $F \subseteq A$ and $|F \cap f^{-1}(b)| = 1$ for all $b \in B$. We may then define an injection $g : B \to A$ by

$$g(b) = \text{the unique element of } F \cap f^{-1}(b)$$

for all $b \in B$.