MATH 430: FORMAL LOGIC
SPRING 2018
HOMEWORK 5

Throughout this homework "theory" means "set of first order formulas".

1. Let φ be a first order formula and T a theory. Suppose φ is true in every infinite model of T. Show there is a natural number n so that φ is true in all finite models of T with size at least n.

2. Use the Compactness Theorem to prove Gödel's Completeness Theorem. Namely:

(a) Compactness: For all theories Σ and formulas τ, if $\Sigma \models \tau$ then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$.

(b) Completeness: For all theories Σ, if Σ is consistent, then Σ is satisfiable.

Assuming (a), prove (b).

3. In the language with one binary relation, $\mathcal{L} = \{\leq\}$, let Σ_{DLO} be the theory of dense linear orders without endpoints. Show Σ_{DLO} is complete: For all sentences τ, either $\Sigma_{\text{DLO}} \vdash \tau$ or $\Sigma_{\text{DLO}} \vdash \neg\tau$.

4. Show the following.

(a) There is no increasing sequence $f : \omega_1 \to \mathbb{R}$.

(b) For all $\alpha \in \omega_1$, there is an increasing sequence $f_\alpha : \alpha \to \mathbb{R}$.

5. Let x be a set. Let

$$a = \{\beta \mid \beta \text{ is an ordinal, and there is an injection } g : \beta \to x\}.$$

(a) Argue, appealing to the axioms of ZF, that a is a set.

(b) Show a is a non-zero ordinal.

(c) Show a is the least ordinal so that $a \not\preceq x$.