MATH 430: FORMAL LOGIC
SPRING 2018
SOLUTIONS TO HOMEWORK 5

1. Let φ be a first order formula and T a theory. Suppose φ is true in every infinite model of T. Show there is a natural number n so that φ is true in all finite models of T with size at least n.
Solution. Suppose not. Then for every natural number n, there is a model of T of size at least n in which φ does not hold. That is, $T \cup \{\neg\varphi\}$ has arbitrarily large finite models. By a theorem from class, $T \cup \{\neg\varphi\}$ has an infinite model. But this contradicts our assumption that φ is true in every infinite model of T.

2. Use the Compactness Theorem to prove Gödel's Completeness Theorem. Namely:

(a) Compactness: For all theories Σ and formulas τ, if $\Sigma \models \tau$ then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$.
(b) Completeness: For all theories Σ, if Σ is consistent, then Σ is satisfiable.

Assuming (a), prove (b).

CORRECTION: Furthermore assume the following version of Completeness: If φ is true in every structure, then φ is provable from the empty set of formulas.
Solution. We must prove (b). Let us prove the contrapositive. Suppose Σ is unsatisfiable. Then we have, vacuously, that

$$\Sigma \models \tau$$

for all formulas τ; in particular, we may take τ to be $\varphi \wedge \neg\varphi$ for some formulas φ. Now by compactness, there is a finite subset $\Sigma_0 \subseteq \Sigma$ so that $\Sigma_0 \models \tau$. Let ρ be the conjunction of all the formulas in Σ_0. Then we have $\models \rho \to \tau$, that is, $\rho \to \tau$ is true in all structures.

Now by our CORRECTION, we have that $\vdash \rho \to \tau$, that is, $\rho \to \tau$ is provable from the empty set. In particular, since $\Sigma_0 \vdash \rho$, we have by modus ponens that $\Sigma \vdash \tau$. Since τ was the negation of a tautology, we've shown Σ is inconsistent, so that (b) holds as desired.

3. In the language with one binary relation, $\mathcal{L} = \{\leq\}$, let Σ_{DLO} be the theory of dense linear orders without endpoints. Show Σ_{DLO} is complete: For all sentences τ, either $\Sigma_{\mathrm{DLO}} \vdash \tau$ or $\Sigma_{\mathrm{DLO}} \vdash \neg\tau$.
Solution. Suppose for a contradiction that the theory is not complete. That is, for some sentence τ, Σ_{DLO} does not prove either τ or $\neg\tau$. Then (by Homework 2 problem 1(a)) we have that $\Sigma_{\mathrm{DLO}} \cup \{\tau\}$ and $\Sigma_{\mathrm{DLO}} \cup \{\neg\tau\}$ are both consistent.

By Gödel's Completeness Theorem, there are (countable) structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ satisfying $\Sigma_{\mathrm{DLO}} \cup \{\tau\}$ and $\Sigma_{\mathrm{DLO}} \cup \{\neg\tau\}$ respectively. Now $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are countable dense linear orders without endpoints, so by Cantor's Theorem, these are isomorphic. But isomorphic structures are elementarily equivalent, contradicting the fact that $\mathfrak{A}_1 \models \tau$ and $\mathfrak{A}_2 \models \neg\tau$. We have our contradiction, so this completes the proof.

5. Show the following.

(a) There is no increasing sequence $f : \omega_1 \to \mathbb{R}$.
(b) For all $\alpha \in \omega_1$, there is an increasing sequence $f_\alpha : \alpha \to \mathbb{R}$.

Solution. (a) Suppose towards a contradiction that $f : \omega_1 \to \mathbb{R}$ is increasing. Then for all $\alpha < \omega_1$, $f(\alpha) < f(\alpha + 1)$. So by density of the rationals, we have some rational q_α with $f(\alpha) < q_\alpha < f(\alpha + 1)$. But then $\{q_\alpha \mid \alpha < \omega_1\}$ is an uncountable set of rational numbers, a contradiction.

(b) Solution 1: Transfinite induction. We show by induction on α that for all $\alpha < \omega_1$ there is an injection $f_\alpha : \alpha \to \mathbb{R}$.

For this, first fix a family of increasing bijections $g_n : \mathbb{R} \to (n, n + 1)$ for all natural numbers $\mathbb{R}$; for instance, we could let $g_n(x) = \arctan(x)/\pi + n + 1/2$.

We have that the empty function is (vacuously) increasing, so $f_0 = \emptyset$.

Suppose we have defined increasing functions f_β for all $\beta < \alpha$. If $\alpha = \gamma + 1$ for some γ, then we can let

$$f_\alpha(\xi) = \begin{cases} g_0(f_\gamma(\xi)) & \text{if } \xi < \gamma \\ 1 & \text{if } \xi = \gamma. \end{cases}$$

Since g_0 is monotonically increasing, and clearly $f_\alpha[\gamma] \subset (0, 1)$, we have that f_α is increasing.

Now suppose α is not a successor. Here is the idea: every $\gamma < \alpha$ is in the domain of some f_α already. We will piece together functions f_{γ_n} for $n \in \mathbb{N}$, where the γ_n are unbounded in α, to define an injection $f_\alpha : \alpha \to \mathbb{R}$.

Claim: There is an increasing function $h : \omega \to \alpha$ so that $\alpha = \sup_n h(n)$.

Proof of claim: Since α is countable and infinite, let $d : \omega \to \alpha$ be a bijection. Inductively define:

$$\begin{aligned} h(0) &= 0, \\ h(n + 1) &= \max\{h(n), d(0), d(1), \ldots, d(n)\} + 1 \end{aligned}$$

Note $h(n) \in \alpha$ for all n since each $d(n)$ is, and we are assuming $\alpha \neq \gamma + 1$ for any ordinal γ. Clearly it is increasing by construction, and since every $\gamma < \alpha$ is $d(n)$ for some n and $d(n) < h_\alpha(n + 1)$, we have that h is unbounded in α, that is, $\alpha = \sup_n h(n)$ as claimed.

The point of this claim is that we have an "increasing" partition of α in order type ω. So we can put:

$$f_\alpha(\gamma) = g_n(f_{h(n)}(\gamma)$$

where n is least so that $\gamma < h(n)$. I leave it to you to check that f_α is increasing.

Solution 2: The better way.

Suppose α is infinite and countable. Then there is a bijection $d : \mathbb{N} \to \alpha$. Define f_α by induction on $n \in \omega$: Let $f_\alpha(d(0)) = 0$. Having defined $f_\alpha \restriction \{d(0), \ldots, d(n)\}$, define $f_\alpha(d(n + 1))$ as follows:

If $d(n + 1) < d(i)$ for all $i \leq n$, let $f_\alpha(d(n + 1)) = \min\{f_\alpha(d(0)), \ldots, f_\alpha(d(n))\} - 1$.

If $d(n + 1) > d(i)$ for all $i \leq n$, let $f_\alpha(d(n + 1)) = \min\{f_\alpha(d(0)), \ldots, f_\alpha(d(n))\} + 1$.

If neither is the case, then let $i, j \leq n$ be such that $d(i) < d(n + 1) < d(j)$, and for no $k \leq n$ do we have $d(i) < d(k) < d(j)$ (that is, $d(i)$ and $d(j)$ are the immediate predecessor and successor, respectively, of $d(n + 1)$ in $\{d(0), \ldots, d(n + 1)\}$. Then let $f_\alpha(n + 1) = \frac{1}{2}[f_\alpha(d(i)) + f_\alpha(d(j))]$.

Then f_α is increasing, and in fact $f_\alpha : \alpha \to \mathbb{Q}$.

6. Let x be a set. Let

$$a = \{\beta \mid \beta \text{ is an ordinal, and there is an injection } g : \beta \to x\}.$$

(a) Argue, appealing to the axioms of ZF, that a is a set.
(b) Show a is a non-zero ordinal.
(c) Show a is the least ordinal so that $a \not\preceq x$.

Solution. (a) By Powerset, $\mathcal{P}(x \times x)$, the set of binary relations on x, exists. By Separation, $\mathrm{WO}_x = \{R \in \mathcal{P}(x \times x) \mid R \text{ is a wellorder of a subset of } x\}$ is a set. For each $R \in \mathrm{WO}_x$, there is a unique ordinal α_R and isomorphism $f_R : \mathrm{dom}(R) \to \alpha_R$. Note that the correspondence $R \mapsto \alpha_R$ is definable. Then by Replacement,

$$\{\alpha_R \mid R \in \mathrm{WO}_x\}$$

is a set, and this set is precisely a.

(b) $0 \in a$ since the empty function is injective, hence $a \neq 0$. a is transitive because if $f : \alpha \to x$ injective, then so is $f \restriction \beta$ for $\beta < \alpha$ (note $\beta \subseteq \alpha$). Since a is a transitive set of ordinals, it is an ordinal.

(c) Clearly, for every $\beta \in a$, we have by definition $\beta \preceq x$. We just need $a \not\preceq x$.

If we had $a \preceq x$, then by (b), we'd have a is an ordinal that injects into x. But then $a \in a$, contradicting that a is an ordinal.